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A classification of 2D fermionic and bosonic topological orders

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The string-net approach by Levin and Wen, and the local unitary transformation approach by Chen, Gu, and Wen, provide ways to systematically describe topological orders in 2D boson systems. The two approaches reveal that the mathematical framework for topological order is closely related to (projective) tensor category theory. In this paper, we generalize those systematic descriptions of topological orders to 2D fermion systems. We find that a large class of topological orders in 2D fermion systems can be systematically characterized by the following set of data $(N_{ijk}, N^f_{ijk}, F^{ijm,\alpha\beta,a}_{jkn,\chi\delta,b}, O^{jk,\alpha\beta}_{i,a})$ that satisfy a set of non-linear algebraic equations. The exactly soluble Hamiltonians can be constructed from the above data on any lattices to realize the corresponding topological orders. Since topological orders in 2D boson systems described by the string-net approach have a deep relationship with the Turaev-Viro states summation invariance of 3-manifolds, it is natural to expect our classification of topological orders in 2D fermionic systems will generalize the famous Turaev-Viro invariance to its fermionic version. When $N^f_{ijk} = 0$, our results provide a classification of bosonic topological orders that generalizes the string-net classification.

CONTENTS

I. INTRODUCTION

Understanding phases of matter is one of the central problems in condensed matter physics. Landau symmetry breaking theory,[1,2] as a systematic theory of phases and phase transitions, becomes a corner stone of condensed matter theory. However, at zero temperature,

the symmetry breaking states described by local order parameters are basically direct product states. It is hard to believe that various direct product states can describe all possible quantum phases of matter.

Based on adiabatic evolution, one can show that gapped quantum phases at zero temperature correspond to the equivalence classes of local unitary (LU) transformations generated by finite-time evolutions of local hermitian operators $\tilde{H}(\tau)$:[3–6]

$$|\Phi\rangle \sim |\Phi'\rangle \quad \text{iff } |\Phi'\rangle = \mathcal{T} e^{\,i \int d\tau \tilde{H}(\tau)} |\Phi\rangle. \tag{1}$$

It turns out that there are many gapped quantum states that cannot be transformed into direct product states through LU transformations. Those states are said to have a long range entanglement. Thus, the equivalence classes of LU transformations, and hence the quantum phases of matter, are much richer than direct product states and much richer than what the symmetry breaking theory can describe. Different patterns of long range entanglement correspond different quantum phases that are beyond the symmetry-breaking/order-parameter description[7] and direct-product-state description. The patterns of long range entanglement really correspond to the topological orders[8,9] that describe the new kind of orders in quantum spin liquids and quantum Hall states.[10–18]

In absence of translation symmetry, the above LU transformation can be expressed as a quantum circuit, which corresponds to a discretized LU transformation. The discretized LU transformation is more convenient to use. The gapped quantum phases can be more effectively studied and even classified through the discretized LU transformations.[6,19–21]

After discovering more and more kinds of topological orders, it becomes important to gain a deeper understanding of topological order under a certain mathematical framework. We know that symmetry breaking orders can be understood systematically under the mathematical framework of group theory. Can topological orders be also understood under some mathematical framework? From the systematic construction of topologically ordered states based on string-nets[19] and the systematic description of non-Abelian satistics[22], it appears that tensor category theory may provide the underlying mathematical framework for topological orders.[23]

However, the string-net and the LU transformation approaches[6,19] only provide a systematic understanding for topological orders in qubit systems (*ie* quantum spin systems or local boson systems). Fermion systems can also have non trivial topological orders. In this paper, we will introduce a systematic theory for topological orders in interacting fermion systems (with interacting boson systems as special cases). Our approach is based on the LU transformations generated by local hermitian operators that contain even number of fermion operators. It allows us to classify and construct a large class of topological orders in fermion systems. The mathematical framework developed here may be related to the theory of enriched categories,[24] which can be viewed as a generalization of the standard tensor category theory[25,26]

To gain a systematic understanding of topological order in fermion systems, we first need a way to label those fermionic topological orders. In this paper, we show that a large class of fermionic topological orders (which include bosonic topological orders as special cases) can be labeled by a set of tensors: $(N_{ijk}, N^f_{ijk}, F^{ijm,\alpha\beta,a}_{jkn,\chi\delta,b}, O^{jk,\alpha\beta}_{i,a})$. Certainly, not every set of tensors corresponds to a valid fermionic topological order. We show that only the tensors that satisfy a set of non-linear equations correspond to valid fermionic topological orders. The set of non-linear equations obtained here is a generalization of the non-linear equations (such as the pentagon identity) in a tensor category theory. So our approach is a generalization of tensor category theory and the string-net approach for bosonic topological orders. We like to point out that the framework developed here not only lead to a classification of fermionic topological orders, it also lead to a more general classification of bosonic topological orders than the string-net and the related approaches.[6,19]

From a set of the data $(N_{ijk}, N^f_{ijk}, F^{ijm,\alpha\beta,a}_{jkn,\chi\delta,b}, O^{jk,\alpha\beta}_{i,a})$, we can obtain an Hamiltonian as a sum of projectors. We believe that the Hamiltonian is unfrustrated. Its zero-energy ground state realizes the fermionic/bosonic topological order described by the data. For a subclass of the topological order, the data satisfies a set of non-linear equation which are not projective. The ideal Hamiltonians for such a subclass of topological orders are sums of commuting projectors, and are thus exactly soluble.

In this paper, we will concentrate on bosonic and fermionic topological orders in 2D interacting systems. We like to mention that recently some significant progresses have been made for bosonic and fermionic (symmetry protected) topological orders in 1D interacting systems: a complete classification is obtained,[27–29] which generalizes some partial results obtained earlier.[20,30–34]

In section II, we give a careful discussion on what is a local fermion system. In section III, we introduce fermionic local unitary transformations. We then use fermionic local unitary transformations to define quantum phases for local fermion systems and fermionic topological orders. In section IV, we use fermionic local unitary transformations to define a wave function renormalization flow for local fermion systems. In section V, we discuss the fixed points of the wave function renormalization flow, and use those fixed points to classify a large class of fermionic (and bosonic) topological orders. In section VI, we present a graphic representation of some of our results. In section VII, we comment on the relation to categorical framework. In section VIII, we give a few simple examples. In appendix A, we discuss the fermionic structure of the support space. In appendix B, we derive the ideal Hamiltonian from the data that characterized the fermionic/bosonic topological orders. In appendix C, we describe an even more general classification of fermionic topological orders.

II. LOCAL FERMION SYSTEMS

Local boson systems (*ie* local qubit systems) and local fermion systems have some fundamental differences. To reveal those differences, in this section, we are going to define local fermion systems carefully. To contrast local fermion systems with local boson systems, let us first review the definition of local boson systems.

A. Local bosonic operators and bosonic states in a local boson model

A local boson quantum model is defined through its Hilbert space V and its local boson Hamiltonian H. The Hilbert space V of a local boson quantum model has a structure $V = \otimes V_{\boldsymbol{i}}$ where $V_{\boldsymbol{i}}$ is the local Hilbert space on the site $\boldsymbol{i}$. A local bosonic operator is defined as an operator that act within the local Hilbert space $V_{\boldsymbol{i}}$, or as a finite product of local bosonic operators acting on nearby sites. A local boson Hamiltonian H is a sum of local bosonic operators. The ground state of a local boson Hamiltonian H is called a bosonic state.

B. Local fermionic operators and fermionic states in a local fermion model

Now, let us try to define local fermion systems. A *local fermion quantum model* is also defined through its Hilbert space V and its local fermion Hamiltonian H_f. First let us introduce *fermion operator* $c^\alpha_{\boldsymbol{i}}$ at site $\boldsymbol{i}$ as operators that satisfy the anticommutation relation

$$c^\alpha_{\boldsymbol{i}} c^\beta_{\boldsymbol{j}} = -c^\beta_{\boldsymbol{j}} c^\alpha_{\boldsymbol{i}}, \qquad c^\alpha_{\boldsymbol{i}} (c^\beta_{\boldsymbol{j}})^\dagger = -(c^\beta_{\boldsymbol{j}})^\dagger c^\alpha_{\boldsymbol{i}}, \tag{2}$$

for all $\boldsymbol{i} \neq \boldsymbol{j}$ and all values of the α, β indices. We also say $c^\alpha_{\boldsymbol{i}}$ acts on the site $\boldsymbol{i}$. The fermion Hilbert space V is the space generated by the fermion operators and their hermitian conjugate:

$$V = \{[c^\beta_{\boldsymbol{j}} (c^\alpha_{\boldsymbol{i}})^\dagger ...]|0\rangle\}. \tag{3}$$

Due to the anticommutation relation (2), V has a form $V = \otimes V_{\boldsymbol{i}}$ where $V_{\boldsymbol{i}}$ is the local Hilbert space on the site $\boldsymbol{i}$. We see that the total Hilbert space of a fermion system has the same structure as a local boson system.

Using the Hilbert space $V = \otimes V_{\boldsymbol{i}}$, an explicit representation of the fermion operator $c^\alpha_{\boldsymbol{i}}$ can be obtained. First, each local Hilbert space can be splitted as $V_{\boldsymbol{i}} = V^0_{\boldsymbol{i}} \oplus V^1_{\boldsymbol{i}}$. We also choose an ordering of the site label $\boldsymbol{i}$. Then $c^\alpha_{\boldsymbol{i}}$ has the following matrix representation

$$\begin{aligned} c^\alpha_{\boldsymbol{i}} &= C^\alpha_{\boldsymbol{i}} \prod_{\boldsymbol{j}<\boldsymbol{i}} \Sigma^3_{\boldsymbol{j}}, \\ C^\alpha_{\boldsymbol{i}} &= \begin{pmatrix} 0 & A^\alpha_{\boldsymbol{i}} \\ B^\alpha_{\boldsymbol{i}} & 0 \end{pmatrix}, \quad \Sigma^3_{\boldsymbol{i}} = \begin{pmatrix} I^0_{\boldsymbol{i}} & 0 \\ 0 & -I^1_{\boldsymbol{i}} \end{pmatrix}, \end{aligned} \tag{4}$$

where $I^0_{\boldsymbol{i}}$ is the identity matrix acting in the space $V^0_{\boldsymbol{i}}$ and $I^1_{\boldsymbol{i}}$ is the identity matrix acting in the space $V^1_{\boldsymbol{i}}$. The matrix $C^\alpha_{\boldsymbol{i}}$ maps a state in $V^0_{\boldsymbol{i}}$ to a state in $V^1_{\boldsymbol{i}}$, and vice versa. We note that

$$C^\alpha_{\boldsymbol{i}} \Sigma^3_{\boldsymbol{i}} = -\Sigma^3_{\boldsymbol{i}} C^\alpha_{\boldsymbol{i}}. \tag{5}$$

We see that a fermion operator is not a local bosonic operator. The product of an odd number of fermion operators and any number of local bosonic operators on nearby sites is called a *local fermionic operator*.

Let us write the eigenvalue of $\Sigma^3_{\boldsymbol{i}}$ as $(-)^{s_{\boldsymbol{i}}}$. The states in $V^0_{\boldsymbol{i}}$ have $s_{\boldsymbol{i}} = 0$ and are called bosonic states. The states in $V^1_{\boldsymbol{i}}$ have $s_{\boldsymbol{i}} = 1$ and are called fermionic states. We can view $s_{\boldsymbol{i}}$ as the fermion number on site $\boldsymbol{i}$.

A *local fermion Hamiltonian* H_f is a sum of terms: $H = \sum_P O_P$, where $\sum_P$ sums over a set of regions. Each term O_P is a product of an even number of local fermionic operators and any number of local bosonic operators on a finite region P. Such kind of terms is called *pseudo-local bosonic operator* acting on the region. In other words, a local fermion Hamiltonian is a sum of pseudo-local bosonic operators. The ground state of a local fermion Hamiltonian H_f is called a *fermionic state*.

Note that, beyond 1D, a pseudo-local bosonic operator is in general not a local bosonic operator. So a local fermion Hamiltonian H_f (beyond 1D) in general is not a local boson Hamiltonian defined in the last subsection. In this sense, a local boson system and a local fermion system are fundamentally different despite they have the same Hilbert space. When viewed as a boson system, a local fermion Hamiltonian corresponds to a non-local boson Hamiltonian (beyond 1D). Thus classifying the quantum phases of local fermion systems corresponds to classifying the quantum phases of a particular kind of non-local boson systems.

III. FERMIONIC LOCAL UNITARY TRANSFORMATION AND TOPOLOGICAL PHASES OF FERMION SYSTEMS

Similar to the local boson systems, the finite-time evolution generated by a local fermion Hamiltonian defines an equivalence relation between gapped fermionic states:

$$|\psi(1)\rangle \sim |\psi(0)\rangle \text{ iff } |\psi(1)\rangle = \mathcal{T}[\mathrm{e}^{\mathrm{i}\int_0^1 dg\, \tilde H_f(g)}]|\psi(0)\rangle \tag{6}$$

where $\mathcal{T}$ is the path-ordering operator and $\tilde H(g) = \sum_{\boldsymbol{i}} O_{\boldsymbol{i}}(g)$ is a local fermion Hamiltonian (*ie* $O_{\boldsymbol{i}}(g)$ is a pseudo-local bosonic operator which is a product of even local fermionic operators). We will call $\mathcal{T}[\mathrm{e}^{\mathrm{i}\int_0^1 dg\, \tilde H_f(g)}]$ a fermion local unitary (fLU) evolution. We believe that the equivalence classes of such an equivalence relation is the universality classes of the gapped quantum phases of fermion systems.

The finite-time fLU evolutions introduced here is closely related to *fermion quantum circuits with finite*

depth. To define fermion quantum circuits, let us introduce piece-wise fermion local unitary operators. A piece-wise fermion local unitary operator has a form $U_{pwl} = \prod_{i} \mathrm{e}^{\mathrm{i} H_f(\boldsymbol{i})}$, where $\{H_f(\boldsymbol{i})\}$ is a Hermitian operator which is a pseudo-local bosonic operator that acts on a region labeled by $\boldsymbol{i}$. Note that regions labeled by different $\boldsymbol{i}$'s are not overlapping. $U_{\boldsymbol{i}} = \mathrm{e}^{\mathrm{i} H_f(\boldsymbol{i})}$ is called a fermion unitary operator. The size of each region is less than some finite number l. The unitary operator U_{pwl} defined in this way is called a fermion piece-wise local unitary operator with range l. A fermion quantum circuit with depth M is given by the product of M fermion piece-wise local unitary operators: $U^M_{circ} = U^{(1)}_{pwl} U^{(2)}_{pwl} \cdots U^{(M)}_{pwl}$. We believe that finite time fLU evolution can be simulated with a constant depth fermion quantum circuit and vice versa. Therefore, the equivalence relation eqn. (6) can be equivalently stated in terms of constant depth fermion quantum circuits:

$$|\psi(1)\rangle \sim |\psi(0)\rangle \text{ iff } |\psi(1)\rangle = U^M_{circ}|\psi(0)\rangle \tag{7}$$

where M is a constant independent of system size. Because of their equivalence, we will use the term "fermion Local Unitary Transformation" to refer to both fermion local unitary evolution and constant depth fermion quantum circuit in general.

Just like boson systems, the equivalence classes of fermionic local unitary transformations correspond to the universality classes that define phases of matter. Since here we do not include any symmetry, the equivalence classes actually correspond to topologically ordered phases. Such topologically ordered phases will be called fermionic topologically ordered phases.

IV. FERMIONIC LOCAL UNITARY TRANSFORMATION AND WAVE FUNCTION RENORMALIZATION

After defining the fermionic topological order as the equivalence classes of many-body wave functions under fLU transformations, we like to use the fLU transformations, or more precisely the generalized fermion local unitary (gfLU) transformation, to define a wave function renormalization procedure. The wave function renormalization can remove the non-universal short-range entanglement and make generic complicated wave functions to flow to some very simple fixed-point wave functions. The simple fixed-point wave functions can help us to classify fermionic topological orders.

Let us first define the gfLU transformation U_g more carefully. Consider a state $|\psi\rangle$. Let ρ_A be the entanglement density matrix of $|\psi\rangle$ in region A. ρ_A may act in a subspace of the Hilbert space in region A. The subspace is called the support space $\tilde{V}_A$ of region A (see Fig. 1(a)). Let $|\tilde{\phi}_{\tilde{i}}\rangle$ be a basis of this support space $\tilde{V}_A$, and $|\phi_i\rangle$ be a basis of the full Hilbert space V_A of region A. The gfLU transformation U_g is the projection from the full Hilbert space V_A to the support space $\tilde{V}_A$. So up to some unitary transformations, U_g is a hermitian projection operator:

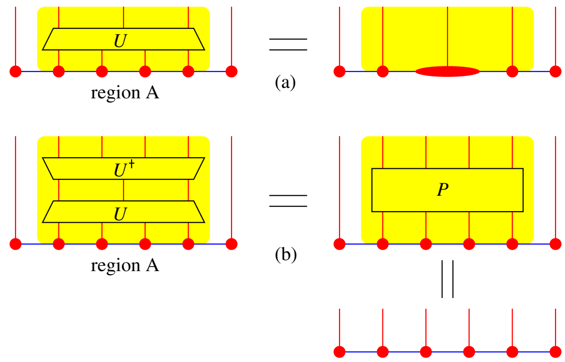


FIG. 1. (Color online) (a) A fermion local unitary (gfLU) transformation U_g acts in region A of a fermionic state $|\psi\rangle$ are formed by bosonic and fermionic operators in the region A. U_g always contain even numbers of fermionic operators. (b) $U_g^\dagger U_g = P$ is a projector, whose action does not change the state $|\psi\rangle$.

$$\begin{aligned} U_g &= U_1 P_g U_2, \qquad P_g^2 = P_g, \qquad P_g^\dagger = P_g, \\ U_1^\dagger U_1 &= 1, \qquad U_2^\dagger U_2 = 1. \end{aligned} \tag{8}$$

The matrix elements of U_g are given by $\langle\tilde{\phi}_{\tilde{a}}|\phi_i\rangle$. We will call such a gfLU transformation a primitive gfLU transformation. A generic gfLU transformation is a product of several primitive gfLU transformations which may contain several hermitian projectors and unitary transformations, for example, $U_g = U_1 P_g U_2 P'_g U_3$.

To understand the fermionic structure of U_g, we note that the support space $\tilde{V}_A$ has a structure $\tilde{V}_A = \tilde{V}^0_A \oplus \tilde{V}^1_A$ (see appendix A), where $\tilde{V}^0_A$ has even numbers of fermions and $\tilde{V}^1_A$ has odd numbers of fermions. This means that U_g contains only even numbers of fermionic operators (*ie* U_g is a pseudo-local bosonic operator).

We also regard the inverse of U_g, $U_g^\dagger$, as a gfLU transformation. A fLU transformation is viewed as a special case of gfLU transformations where the degrees of freedom are not changed. Clearly $U_g^\dagger U_g = P$ and $U_g U_g^\dagger = P'$ are two hermitian projectors. The action of P does not change the state $|\psi\rangle$ (see Fig. 1(b)). Thus despite the degrees of freedom can be reduced under the gfLU transformations, no quantum information of the state $|\psi\rangle$ is lost under the gfLU transformations.

We note that the gfLU transformations can map one wave function to another wave function with fewer degrees of freedom. Thus it can be viewed as a wave function renormalization group flow. If the wave function renormalization leads to fixed-point wave functions, then those fixed-point wave functions can be much simpler, which can provide an efficient or even one-to-one labeling scheme of fermionic topological orders.

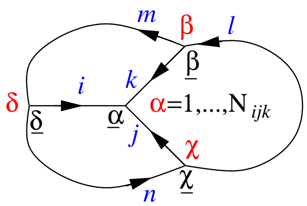


FIG. 2. (Color online) A quantum state on a vertex-labeled graph G. The four vertices are labeled by $\underline{\alpha}$, $\underline{\beta}$, $\underline{\chi}$, and $\underline{\delta}$. For example, $\underline{\alpha} = 1$, $\underline{\beta} = 2$, $\underline{\chi} = 3$, $\underline{\delta} = 4$, correspond to one vertex labeling scheme. States on each edge are labeled by $l = 0, ..., N$. If the three edges of a vertex are in the states i, j, and k respectively, then the vertex has N_{ijk} states. The states on the $\underline{\alpha}$-vertex are labeled by $\alpha = 1, ..., N_{ijk}$. Note the orientation of the edges are point towards to vertex. Also note that $i \to j \to k$ runs anti-clockwise.

FIG. 3. The mapping $i \to i^*$ corresponds to the reverse of the orientation of the edge

V. WAVE FUNCTION RENORMALIZATION AND A CLASSIFICATION OF FERMIONIC TOPOLOGICAL ORDERS

As an application of the above fermionic wave function renormalization, in this section, we will study the structure of fixed-point wave functions under the wave function renormalization. This will lead to a classification of fermionic topological orders.

A. Quantum state on a graph

Since the wave function renormalization may change the lattice structure, we will consider quantum state defined on a generic trivalent graph G: Each edge has $N+1$ states, labeled by $i = 0, ..., N$ (see Fig. 2). We assume that the index i on the edge admits a one-to-one mapping $*$: $i \to i^*$ that satisfies $(i^*)^* = i$. As a result, the edges of the graph are oriented. The mapping $i \to i^*$ corresponds to the reverse of the orientation of the edge (see Fig. 3). Each vertex also has physical states. The number of the states depends on the states on the connected edges and they are labeled by $\alpha = 1, ..., N_{ijk}$ (see Fig. 2). Here we assume that

$$N_{ijk} = N_{jki}. \tag{9}$$

Despite the similar look between α index and $\underline{\alpha}$ index, the two indices are very different. $\underline{\alpha}$ index labels the vertices while α index labels the state on a vertex. In this paper, we very often use α to label states on vertex $\underline{\alpha}$.

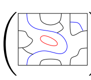
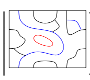
The states on the edge are always bosonic. However, the states on the vertices may be fermionic. We introduce $s_{ijk}(\alpha)$ to indicate whether a vertex state labeled by α is bosonic or fermionic: $s_{ijk}(\alpha) = 0$ if the α-state is bosonic and $s_{ijk}(\alpha) = 1$ if the α-state is fermionic. Here the vertex connects to three edges i, j, and k (see Fig. 2). The function $s_{ijk}(\alpha)$ satisfies

$$s_{ijk}(\alpha) = s_{kij}(\alpha). \tag{10}$$

Only a fermionic operator can map a bosonic state to a fermionic state and vice versa. Let N^b_{ijk} (N^f_{ijk}) be the number of bosonic (fermionic) state on the vertex that connect to three edges with states $|i\rangle$, $|j\rangle$, and $|k\rangle$. We also have

$$N^b_{ijk} = N^b_{jki}, \qquad N^f_{ijk} = N^f_{jki}. \tag{11}$$

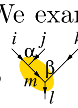
Each graph with a given $\alpha, \beta, ...i, j, ...$ labeling (see Fig. 2) corresponds to a state and all such labeled graphs form an orthonormal basis. Our fixed-point state is a superposition of those basis states

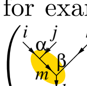
$|\psi_{\text{fix}}\rangle = \sum_{\text{all conf.}} \psi_{\text{fix}}\Big($  $\Big) \Big|$  $\Big\rangle .$ (12)

In the string-net approach, we made a very strong assumption that the above graphic states on two graphs are the same if the two graphs have the same topology. However, since different vertices and edges are really distinct, a generic graph state does not have such an topological invariance. To consider more general states, in this paper, we would like to weaken such a topological requirement. We will consider vertex-labeled graphs (v-graphs) where each vertex is assigned an index $\underline{\alpha}$. Two v-graphs are said to be topologically the same if one graph can be continuously deformed into the other in such a way that vertex labeling of the two graphs matches. In this paper, we will consider the graph states that depend only on the topology of the v-graphs. Those states are more general than the graph states that depend only on the topology of the graphs without vertex labeling. Such a generalization is important in obtaining interesting fermionic fixed-point states on graphs.

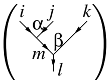
B. The structure of a fixed-point wave function

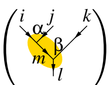
Before describing the wave function renormalization, we examine the structure of entanglement of a fixed point wave function ψ_{fix} on a v-graph. First let us introduce the concept of support space with a fixed boundary state.

We examine the wave function on a patch, for example, 

. The fixed-point wave function $\psi_{\text{fix}}\Big($ 

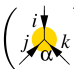
$\Big)$ (only the relevant part of the graph is drawn) can be viewed as a function of α, β, m: $\phi_{ijkl,\Gamma}(\alpha, \beta, m) =$

$\psi_{\rm fix}\Bigg($  $\Bigg)$ if we fix i,j,k,l and the indices on the other part of the graph. (Here the indices on the other part of the graph is summarized by Γ.) As we vary the indices Γ on the other part of the graph (still keep i, j, k, and l fixed), the wave function of α,β,m, $\phi_{ijkl,\Gamma}(\alpha,\beta,m)$, may change. All those $\phi_{ijkl,\Gamma}(\alpha,\beta,m)$ form a linear space of dimension D_{ijkl^*}. D_{ijkl^*} is an important concept that will appear later. We note that the two vertices α and β and the edge m form a region surrounded by the edges i,j,k,l. So we will call the dimension-D_{ijkl^*} space the support space V_{ijkl^*} and D_{ijkl^*} the support dimension for the state $\psi_{\rm fix}$ on the region surrounded by a fixed boundary state i,j,k,l.

We note that in the fixed-point wave function $\psi_{\rm fix}\Bigg($  $\Bigg)$, the number of choices of α,β,m is $N_{ijkl^*}=\sum_{m=0}^N N_{jim^*}N_{kml^*}$. Thus the support dimension D_{ijkl^*} satisfies $D_{ijkl^*}\leq N_{ijkl^*}$. Here we will make an important assumption – the saturation assumption: *The fixed-point wave function saturates the inequality:*

$$D_{ijkl^*}=N_{ijkl^*}\equiv\sum_{m=0}^N N_{jim^*}N_{kml^*}. \tag{13}$$

In general, we will make the similar saturation assumption for any tree graphs. We will see that the entanglement structure described by such a saturation assumption is invariant under the wave function renormalization.

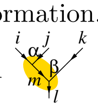
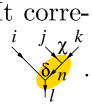
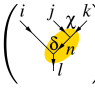
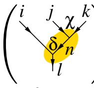
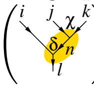
Similarly, we can define D_{ijk} as the support dimension of the $\Phi_{\rm fix}\Bigg($  $\Bigg)$ on a region bounded by links i,j,k. Since the region contains only a single vertex $\underline{\alpha}$, we have $D_{ijk}\leq N_{ijk}$. The saturation assumption requires that

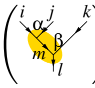
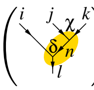
$$D_{ijk}=N_{ijk}. \tag{14}$$

In fact, this is how N_{ijk} is defined.

We note that under the saturation assumption, the structure of the support dimensions for tree graphs is encoded in the N_{ijk} tensor. Here N_{ijk} plays a similar role as the pattern of zeros in a classification of fractional quantum Hall wave functions.[35]

C. The first type of wave function renormalization

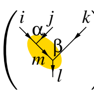
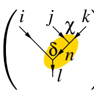
Our wave function renormalization scheme contains two types of renormalization. The first type of renormalization does not change the degrees of freedom and corresponds to a local unitary transformation. It corresponds to locally deform the v-graph  to . (The parts that are not drawn are the same.) The fixed-point wave function on the new v-graph is given by $\psi_{\rm fix}\Bigg($  $\Bigg)$. Again, such a wave function can be viewed as a function of χ,δ,n: $\tilde\phi_{ijkl,\Gamma}(\chi,\delta,n)=\psi_{\rm fix}\Bigg($  $\Bigg)$ if we fix i,j,k,l and the indices on other part of the graph. The support dimension of the state $\psi_{\rm fix}\Bigg($  $\Bigg)$ on the region surrounded by i,j,k,l is $\tilde D_{ijkl^*}$. Again $\tilde D_{ijkl^*}\leq\tilde N_{ijkl^*}$, where $\tilde N_{ijkl^*}\equiv\sum_{n=0}^N N_{kjn^*}N_{l^*ni}$ is number of choices of χ,δ,n. The saturation assumption implies that $\tilde N_{ijkl^*}=\tilde D_{ijkl^*}$.

The two fixed-point wave functions $\psi_{\rm fix}\Bigg($  $\Bigg)$ and $\psi_{\rm fix}\Bigg($  $\Bigg)$ are related via a local unitary transformation. Thus

$$D_{ijkl^*}=\tilde D_{ijkl^*}, \tag{15}$$

which implies

$$\sum_{m=0}^N N_{jim^*}N_{kml^*}=\sum_{n=0}^N N_{kjn^*}N_{l^*ni}. \tag{16}$$

We note that the support space of $\psi_{\rm fix}\Bigg($  $\Bigg)$ and $\psi_{\rm fix}\Bigg($  $\Bigg)$ should have the same number of fermionic states. Thus eqn. (16) can be splitted as

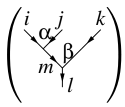
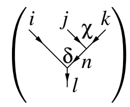
$$\begin{aligned}&\sum_{m=0}^N(N^b_{jim^*}N^b_{kml^*}+N^f_{jim^*}N^f_{kml^*})\\&=\sum_{n=0}^N(N^b_{kjn^*}N^b_{l^*ni}+N^f_{kjn^*}N^f_{l^*ni}),\end{aligned} \tag{17}$$

$$\begin{aligned}&\sum_{m=0}^N(N^b_{jim^*}N^f_{kml^*}+N^f_{jim^*}N^b_{kml^*})\\&=\sum_{n=0}^N(N^b_{kjn^*}N^f_{l^*ni}+N^f_{kjn^*}N^b_{l^*ni}).\end{aligned} \tag{18}$$

We express the above unitary transformation in terms of the tensor $F^{ijm,\alpha\beta,a}_{kln,\chi\delta,b}$, where $a,b=\pm$, $i,j,k,...=0,...,N$, and $\alpha=1,...,N_{ijk}$, *etc* :

$$\phi_{ijkl,\Gamma}(\alpha,\beta,m)\simeq\sum_{n=0}^N\sum_{\chi=1}^{N_{kjn^*}}\sum_{\delta=1}^{N_{nil^*}}F^{ijm,\alpha\beta,\underline{\alpha\beta}}_{kln,\chi\delta,\underline{\chi\delta}}\tilde\phi_{ijkl,\Gamma}(\chi,\delta,n) \tag{19}$$

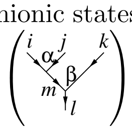
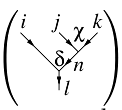
or graphically as

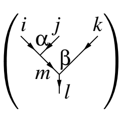
ψ_{fix}  $\simeq \sum_{n\chi\delta} F^{ijm,\alpha\beta,\underline{\alpha\beta}}_{kln,\chi\delta,\underline{\chi\delta}}\, \psi_{\text{fix}}$  . (20)

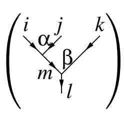
where the vertices carrying the states labeled by $(\alpha,\beta,\chi,\delta)$ are labeled by $(\underline{\alpha},\underline{\beta},\underline{\chi},\underline{\delta})$ (see Fig. 2). Here $\simeq$ means equal up to a constant phase factor. (Note that the total phase of the wave function is unphysical.) We will call such a wave function renormalization step an F-move.

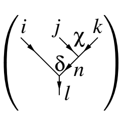
In general, we should write the F-tensor as $F^{ijm,\alpha\beta,\underline{\alpha},\underline{\beta}}_{kln,\chi\delta,\underline{\chi},\underline{\delta}}$ which depends the vertex labels $\underline{\alpha}$, $\underline{\beta}$, $\underline{\chi}$, $\underline{\delta}$. In this paper, we assume that the F-tensor only depends on the sign of $\underline{\beta}-\underline{\alpha}$. It does not depend on how big is the difference $|\underline{\beta}-\underline{\alpha}|$. So the F-tensor can be written as $F^{ijm,\alpha\beta,\underline{\alpha\beta}}_{kln,\chi\delta,\underline{\chi\delta}}$. Here $\underline{\alpha\beta}$ is a function of $\underline{\alpha}$ and $\underline{\beta}$ describing the sign of $\underline{\beta}-\underline{\alpha}$:

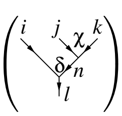
$$\underline{\alpha\beta} = - \ \text{ if } \underline{\alpha} > \underline{\beta}, \quad \underline{\alpha\beta} = + \ \text{ if } \underline{\alpha} < \underline{\beta}. \tag{21}$$

There is a subtlety in eqn. (20). Since some values of α, β, ... indices correspond to fermionic states, the sign of wave function depends on how those fermionic states are ordered. In (20), the wave functions ψ_{fix}  and ψ_{fix}  are obtained by assuming the fermionic states are order in a particular way:

$\Big|\psi_{\text{fix}}$  $\Big\rangle$

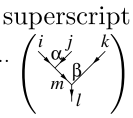
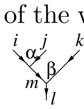
$= \sum \psi_{\text{fix}}^{\underline{\alpha}\,\underline{\beta}\,\underline{\eta}_1\,\underline{\eta}_2\cdots}$  $|\alpha\beta\eta_1\eta_2...\rangle_{\underline{\alpha}\,\underline{\beta}\,\underline{\eta}_1\,\underline{\eta}_2\cdots}$

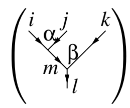
$\Big|\psi_{\text{fix}}$  $\Big\rangle$

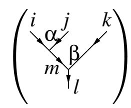
$= \sum \psi_{\text{fix}}^{\underline{\chi}\,\underline{\delta}\,\underline{\eta}_1\,\underline{\eta}_2\cdots}$  $|\chi\delta\eta_1\eta_2...\rangle_{\underline{\chi}\,\underline{\delta}\,\underline{\eta}_1\,\underline{\eta}_2\cdots}$ (22)

where $\sum$ sum over the indices on the vertices and on the edges and η_i are indices on other vertices. Here $|\alpha\beta...\rangle_{\underline{\alpha}\,\underline{\beta}...}$ is a graph state where the $\underline{\alpha}$-vertex is in the $|\alpha\rangle$-state, the $\underline{\beta}$-vertex is in the $|\beta\rangle$-state, *etc* . We note that $|\beta\alpha...\rangle_{\underline{\beta}\,\underline{\alpha}...}$ is also a graph state where the $\underline{\alpha}$-vertex is in the $|\alpha\rangle$-state, the $\underline{\beta}$-vertex is in the $|\beta\rangle$-state. But if the $|\alpha\rangle$-state and the $|\beta\rangle$-state are fermionic (*ie* $s_{jim^*}(\alpha) = s_{kml^*}(\beta) = 1$), $|\alpha\beta...\rangle_{\underline{\alpha}\,\underline{\beta}...}$ and $|\beta\alpha...\rangle_{\underline{\beta}\,\underline{\alpha}...}$ will differ by a sign, since in $|\alpha\beta...\rangle_{\underline{\alpha}\,\underline{\beta}...}$ the fermion on $\underline{\beta}$-vertex is created before the fermion on $\underline{\alpha}$-vertex is created, while in $|\beta\alpha...\rangle_{\underline{\alpha}\,\underline{\beta}...}$ the fermion on $\underline{\alpha}$-vertex is created before the fermion on $\underline{\beta}$-vertex is created. In general we have

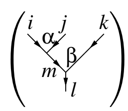
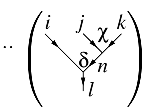
$$|\alpha\beta...\rangle_{\underline{\alpha}\,\underline{\beta}...} = (-)^{s_{jim^*}(\alpha)s_{kml^*}(\beta)}|\beta\alpha...\rangle_{\underline{\beta}\,\underline{\alpha}...}. \tag{23}$$

We see that subscript $\underline{\alpha}\,\underline{\beta}...$ in $|\alpha\beta...\rangle_{\underline{\alpha}\,\underline{\beta}...}$ is important to properly describe such an order dependent sign. Similarly, we must add the superscript in the wave function as well, as in $\psi_{\text{fix}}^{\underline{\alpha}\,\underline{\beta}\,\underline{\eta}_1\,\underline{\eta}_2\cdots}$ , since the amplitude of the wave function depends both one the labeled graph  and the ordering of the vertices $\underline{\alpha}\,\underline{\beta}\,\underline{\eta}_1\,\underline{\eta}_2...$. Such a wave function has the following sign dependence:

$\psi_{\text{fix}}^{\underline{\alpha}\,\underline{\beta}\,\underline{\eta}_1\,\underline{\eta}_2\cdots}$ 

$= (-)^{s_{jim^*}(\alpha)s_{kml^*}(\beta)}\psi_{\text{fix}}^{\underline{\beta}\,\underline{\alpha}\,\underline{\eta}_1\,\underline{\eta}_2\cdots}$  . (24)

Thus eqn. (20) should be more properly written as

$\psi_{\text{fix}}^{\underline{\alpha}\,\underline{\beta}...}$  $\simeq \sum_{n\chi\delta} F^{ijm,\alpha\beta,\underline{\alpha\beta}}_{kln,\chi\delta,\underline{\chi\delta}}\, \psi_{\text{fix}}^{\underline{\chi}\,\underline{\delta}...}$  , (25)

where the superscripts $\underline{\alpha}\,\underline{\beta}...$ and $\underline{\chi}\,\underline{\delta}...$ describing the order of fermionic states are added in the wave function.

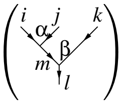
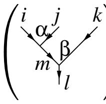
Since the sign of the wave function depends on the ordering of fermionic states, the F-tensor may also depend on the ordering. In this paper, we choose a particular ordering of fermionic states to define the F-tensor as described by $\underline{\alpha}\,\underline{\beta}...$ and $\underline{\chi}\,\underline{\delta}...$ in eqn. (25). In such a canonical ordering, we create a fermion on the $\underline{\beta}$-vertex before we create a fermion on the $\underline{\alpha}$-vertex. Similarly, we create a fermion on the $\underline{\delta}$-vertex before we create a fermion on the $\underline{\chi}$-vertex.

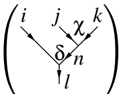
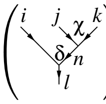
In the following, we would like to introduce one Majorana numbers $\theta_{\underline{\alpha}}$ on each vertex $\underline{\alpha}$ to rewrite (25) in a form that will be valid for any ordering of fermionic states on vertices. The Majorana numbers satisfy

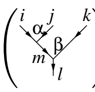
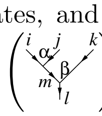
$$\begin{aligned} &\theta^2_{\underline{\alpha}} = 1, \quad \theta_{\underline{\alpha}}\theta_{\underline{\beta}} = -\theta_{\underline{\beta}}\theta_{\underline{\alpha}} \text{ for any } \alpha\neq\beta, \\ &\theta^\dagger_{\underline{\alpha}} = \theta_{\underline{\alpha}}, \quad (\theta_{\underline{\alpha}}...\theta_{\underline{\beta}})^\dagger = \theta_{\underline{\beta}}...\theta_{\underline{\alpha}}. \end{aligned} \tag{26}$$

We introduce the following wave function with Majorana

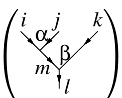
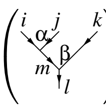
numbers:

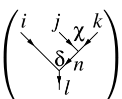
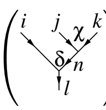
Ψ_{fix} () $= [\theta_{\underline{\alpha}}^{s_{jim^*}(\alpha)}\theta_{\underline{\beta}}^{s_{kml^*}(\beta)}\dots]\psi_{\text{fix}}^{\underline{\alpha}\,\underline{\beta}..}$ ()

Ψ_{fix} () $= [\theta_{\underline{\chi}}^{s_{kjn^*}(\chi)}\theta_{\underline{\delta}}^{s_{nil^*}(\delta)}\dots]\psi_{\text{fix}}^{\underline{\chi}\,\underline{\delta}..}$ () (27)

where the order of the Majorana numbers $\theta_{\underline{\alpha}}\theta_{\underline{\beta}}\dots$ is tied to the order $\underline{\alpha}\,\underline{\beta}..$ in the superscript that describes the order of the fermionic states. We see that, by construction, the sign of Ψ_{fix} () does not depend on the order of the fermionic states, and this is why the Majorana wave function Ψ_{fix} () does not carry the superscript $\underline{\alpha}\,\underline{\beta}\dots$

We like to mention that $(\theta_{\underline{\alpha}}, \theta_{\underline{\beta}})$ and $(\theta_{\underline{\chi}}, \theta_{\underline{\delta}})$ are treated as different Majorana number even when, for example, $\underline{\alpha}$ and $\underline{\chi}$ take the same value. This is because $\underline{\alpha}$ and $\underline{\chi}$ label different vertices regardless if $\underline{\alpha}$ and $\underline{\chi}$ have the same value or not. So a more accurate notation should be

Ψ_{fix} () $= [\theta_{\underline{\alpha}}^{s_{jim^*}(\alpha)}\theta_{\underline{\beta}}^{s_{kml^*}(\beta)}\dots]\psi_{\text{fix}}^{\underline{\alpha}\,\underline{\beta}..}$ ()

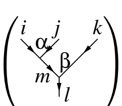
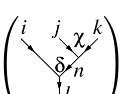
Ψ_{fix} () $= [\tilde\theta_{\underline{\chi}}^{s_{kjn^*}(\chi)}\tilde\theta_{\underline{\delta}}^{s_{nil^*}(\delta)}\dots]\psi_{\text{fix}}^{\underline{\chi}\,\underline{\delta}..}$ (), (28)

where $\theta_{\underline{\alpha}}$ and $\tilde\theta_{\underline{\chi}}$ are different even when $\underline{\alpha} = \underline{\chi}$. But in this paper, we will drop the ˜ and hope that it will not cause any confusions.

Let us introduce the F-tensor with Majorana numbers:

$$\mathcal{F}^{ijm,\alpha\beta,\underline{\alpha\beta}}_{kln,\chi\delta,\underline{\chi\delta}} = \theta_{\underline{\alpha}}^{s_{jim^*}(\alpha)}\theta_{\underline{\beta}}^{s_{kml^*}(\beta)}\theta_{\underline{\delta}}^{s_{nil^*}(\delta)}\theta_{\underline{\chi}}^{s_{kjn^*}(\chi)} F^{ijm,\alpha\beta,\underline{\alpha\beta}}_{kln,\chi\delta,\underline{\chi\delta}} \tag{29}$$

We can rewrite (25) as

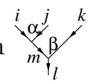
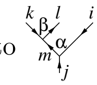
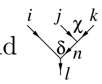
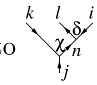
Ψ_{fix} () $\simeq \sum_{n\chi\delta} \mathcal{F}^{ijm,\alpha\beta,\underline{\alpha\beta}}_{kln,\chi\delta,\underline{\chi\delta}} \Psi_{\text{fix}}$ (). (30)

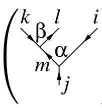
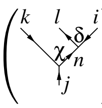
Such an expression is valid for any ordering of the fermion states.

For fixed i, j, k, and l, the matrix $F^{ij,\underline{\alpha\beta}}_{kl,\underline{\chi\delta}}$ with matrix elements $(F^{ij,\underline{\alpha\beta}}_{kl,\underline{\chi\delta}})^{m,\alpha\beta}_{n,\chi\delta} = F^{ijm,\alpha\beta,\underline{\alpha\beta}}_{kln,\chi\delta,\underline{\chi\delta}}$ is a matrix of dimension N_{ijkl^*} (see (16)). Here we require that the mapping $\tilde\phi_{ijkl,\Gamma}(\chi,\delta,n) \to \phi_{ijkl,\Gamma}(\alpha,\beta,m)$ generated by the matrix $F^{ij,\underline{\alpha\beta}}_{kl,\underline{\chi\delta}}$ to be unitary. Since, as we change Γ, $\tilde\phi_{ijkl,\Gamma}(\chi,\delta,n)$ and $\phi_{ijkl,\Gamma}(\alpha,\beta,m)$ span two N_{ijkl^*} dimensional spaces. Thus we require that $F^{ij,\underline{\alpha\beta}}_{kl,\underline{\chi\delta}}$ to be an $N_{ijkl^*} \times N_{ijkl^*}$ unitary matrix

$$\sum_{n\chi\delta} F^{ijm',\alpha'\beta',\underline{\alpha\beta}}_{kln,\chi\delta,\underline{\chi\delta}} (F^{ijm,\alpha\beta,\underline{\alpha\beta}}_{kln,\chi\delta,\underline{\chi\delta}})^* = \delta_{m\alpha\beta,m'\alpha'\beta'}, \tag{31}$$

where $\delta_{m\alpha\beta,m'\alpha'\beta'} = 1$ when $m = m'$, $\alpha = \alpha'$, $\beta = \beta'$, and $\delta_{m\alpha\beta,m'\alpha'\beta'} = 0$ otherwise. This way, the F-move represents a fLU transformation.

We can deform  to , and  to . We see that

Ψ_{fix} () $\simeq \sum_{n\chi\delta} \mathcal{F}^{kl^*m^*,\beta\alpha,\underline{\beta\alpha}}_{ij^*n^*,\delta\chi,\underline{\chi\delta}} \Psi_{\text{fix}}$ (). (32)

Eqn. (30) and eqn. (32) relate the same pair of graphs, and should be the same relations. Thus

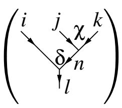
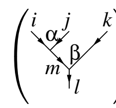
$$\mathcal{F}^{ijm,\alpha\beta,a}_{kln,\chi\delta,b} = \mathcal{F}^{kl^*m^*,\beta\alpha,-a}_{ij^*n^*,\delta\chi,-b} \tag{33}$$

or

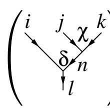
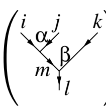
$$\begin{aligned} &F^{ijm,\alpha\beta,a}_{kln,\chi\delta,b} \\ &= (-)^{s_{jim^*}(\alpha)s_{kml^*}(\beta)+s_{kjn^*}(\chi)s_{nil^*}(\delta)} F^{kl^*m^*,\beta\alpha,-a}_{ij^*n^*,\delta\chi,-b} \end{aligned} \tag{34}$$

(where we have used the saturation condition $N_{ijkl} = D_{ijkl}$ and $a = \underline{\alpha\beta} = -\underline{\beta\alpha}$.)

Using the relation (31), we can rewrite eqn. (25) as

$\psi_{\text{fix}}^{\underline{\chi}\,\underline{\delta}..}$ () $\simeq \sum_{m\alpha\beta} (F^{ijm,\alpha\beta}_{kln,\chi\delta})^* \psi_{\text{fix}}^{\underline{\alpha}\,\underline{\beta}..}$ () (35)

or as

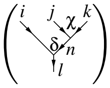
Ψ_{fix} () $\simeq \sum_{m\alpha\beta} (\mathcal{F}^{ijm,\alpha\beta,\underline{\alpha\beta}}_{kln,\chi\delta,\underline{\chi\delta}})^\dagger \Psi_{\text{fix}}$ (), (36)

where

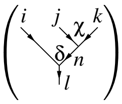
$$\begin{aligned} &(\mathcal{F}^{ijm,\alpha\beta,a}_{kln,\chi\delta,b})^\dagger \\ &= \theta_{\underline{\chi}}^{s_{kjn^*}(\chi)}\theta_{\underline{\delta}}^{s_{nil^*}(\delta)}\theta_{\underline{\beta}}^{s_{kml^*}(\beta)}\theta_{\underline{\alpha}}^{s_{jim^*}(\alpha)} (F^{ijm,\alpha\beta,a}_{kln,\chi\delta,b})^*. \end{aligned} \tag{37}$$

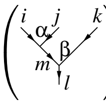
Here we have used

$$\begin{aligned} &(\theta_{\underline{\alpha}}^{s_{jim^*}(\alpha)}\theta_{\underline{\beta}}^{s_{kml^*}(\beta)}\theta_{\underline{\delta}}^{s_{nil^*}(\delta)}\theta_{\underline{\chi}}^{s_{kjn^*}(\chi)})^\dagger \\ &= \theta_{\underline{\chi}}^{s_{kjn^*}(\chi)}\theta_{\underline{\delta}}^{s_{nil^*}(\delta)}\theta_{\underline{\beta}}^{s_{kml^*}(\beta)}\theta_{\underline{\alpha}}^{s_{jim^*}(\alpha)}. \end{aligned} \tag{38}$$

We can also express $\Psi_{\text{fix}}\Bigg($  $\Bigg)$ as

$$\Psi_{\text{fix}}\Bigg(\text{graph} \Bigg) \simeq \sum_{m\alpha\beta} \mathcal{F}^{jkn,\chi\delta,\,\underline{\chi\delta}}_{l^*i^*m^*,\beta\alpha,\,\underline{\beta\alpha}} \Psi_{\text{fix}}\Bigg(\text{graph} \Bigg) \tag{39}$$





using a relabeled eqn. (30). So we have

$$(\mathcal{F}^{ijm,\alpha\beta,a}_{kln,\chi\delta,b})^\dagger \simeq \mathcal{F}^{jkn,\chi\delta,b}_{l^*i^*m^*,\beta\alpha,-a}, \tag{40}$$

Since the total phase of wave function is unphysical, we can adjust the total phase of $\mathcal{F}^{ijm,\alpha\beta,a}_{kln,\chi\delta,b}$ for each choice of $(a,b)=(\pm,\pm)$ independently to make $\simeq$ into $=$:

$$(\mathcal{F}^{ijm,\alpha\beta,a}_{kln,\chi\delta,b})^\dagger = \mathcal{F}^{jkn,\chi\delta,b}_{l^*i^*m^*,\beta\alpha,-a}, \tag{41}$$

or

$$(F^{ijm,\alpha\beta,a}_{kln,\chi\delta,b})^* = (-)^{s_{jim^*}(\alpha)s_{kml^*}(\beta)} F^{jkn,\chi\delta,b}_{l^*i^*m^*,\beta\alpha,-a}. \tag{42}$$

However, the following phase change

$$F^{ijm,\alpha\beta,a}_{kln,\chi\delta,b} \to \mathrm{e}^{\mathrm{i}\,ab\theta} F^{ijm,\alpha\beta,a}_{kln,\chi\delta,b} \tag{43}$$

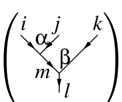
does not affect eqn. (42) and is still not fixed.

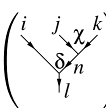
If we apply eqn. (42) twice, we reproduce exactly eqn. (34). Thus eqn. (34) is not independent and can be dropped. If we apply eqn. (42) four times, we get the identity transformation. Applying eqn. (42) once, twice, and three times allow us to express $F^{ijm,\alpha\beta,-}_{kln,\chi\delta,-}$, $F^{ijm,\alpha\beta,-}_{kln,\chi\delta,+}$, $F^{ijm,\alpha\beta,+}_{kln,\chi\delta,+}$, in terms of $F^{ijm,\alpha\beta,+}_{kln,\chi\delta,-}$:

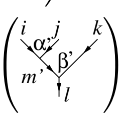
$$\begin{aligned}
F^{ijm,\alpha\beta,+}_{kln,\chi\delta,+} &= (-)^{s_{jim^*}(\alpha)s_{kml^*}(\beta)} (F^{jkn,\chi\delta,+}_{l^*i^*m^*,\beta\alpha,-})^*, \\
F^{ijm,\alpha\beta,-}_{kln,\chi\delta,+} &= (-)^{s_{jim^*}(\alpha)s_{kml^*}(\beta)+s_{kjn^*}(\chi)s_{nil^*}(\delta)} F^{kl^*m^*,\beta\alpha,+}_{ij^*n^*,\delta\chi,-}, \\
F^{ijm,\alpha\beta,-}_{kln,\chi\delta,-} &= (-)^{s_{kjn^*}(\chi)s_{nil^*}(\delta)} (F^{l^*in^*,\delta\chi,+}_{jk^*m,\alpha\beta,-})^*.
\end{aligned} \tag{44}$$

We can use transformation eqn. (30) and eqn. (36) to rearrange the labels of the vertices:

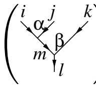
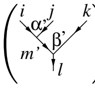
$$\Psi_{\text{fix}}\Bigg(\text{graph} \Bigg) \simeq \sum_{n\chi\delta} \mathcal{F}^{ijm,\alpha\beta,\,\underline{\alpha\beta}}_{kln,\chi\delta,\,\underline{\chi\delta}} \Psi_{\text{fix}}\Bigg(\text{graph} \Bigg) \simeq \sum_{n\chi\delta;m'\alpha'\beta'} \mathcal{F}^{ijm,\alpha\beta,\,\underline{\alpha\beta}}_{kln,\chi\delta,\,\underline{\chi\delta}} (\mathcal{F}^{ijm',\alpha'\beta',\,\underline{\alpha'\beta'}}_{kln,\chi\delta,\,\underline{\chi\delta}})^\dagger \Psi_{\text{fix}}\Bigg(\text{graph} \Bigg). \tag{45}$$







Since $\underline{\alpha\beta}$ and $\underline{\alpha'\beta'}$ can be different, the labeling of the vertices is rearranged.

The above relation between $\Psi_{\text{fix}}\Bigg($  $\Bigg)$ and $\Psi_{\text{fix}}\Bigg($  $\Bigg)$ should be the same regardless if we choose $\underline{\chi\delta}=+$ or $\underline{\chi\delta}=-$. Therefore, we have

$$\sum_{n\chi\delta} \mathcal{F}^{ijm,\alpha\beta,a}_{kln,\chi\delta,+} (\mathcal{F}^{ijm',\alpha'\beta',b}_{kln,\chi\delta,+})^\dagger \simeq \sum_{n\chi\delta} \mathcal{F}^{ijm,\alpha\beta,a}_{kln,\chi\delta,-} (\mathcal{F}^{ijm',\alpha'\beta',b}_{kln,\chi\delta,-})^\dagger, \tag{46}$$

or

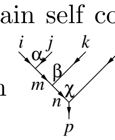
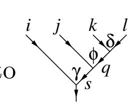
$$\sum_{n\chi\delta} F^{ijm,\alpha\beta,a}_{kln,\chi\delta,+} (F^{ijm',\alpha'\beta',b}_{kln,\chi\delta,+})^* = \mathrm{e}^{\mathrm{i}\,\theta^{ab}_F} \sum_{n\chi\delta} F^{ijm,\alpha\beta,a}_{kln,\chi\delta,-} (F^{ijm',\alpha'\beta',b}_{kln,\chi\delta,-})^*. \tag{47}$$

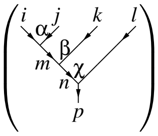
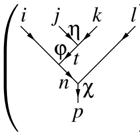
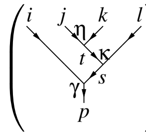
The above conditions on the F-tensor ensure the self consistency of rearranging the labeling of the vertices.

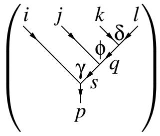
From the graphic representation (25), We note that

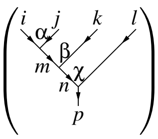
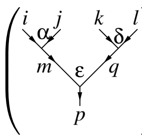
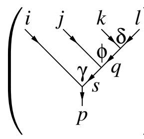
$$\begin{aligned}
&F^{ijm,\alpha\beta,a}_{kln,\chi\delta,b} = 0 \text{ when} \\
&N_{jim^*} < 1 \text{ or } N_{kml^*} < 1 \text{ or } N_{kjn^*} < 1 \text{ or } N_{nil^*} < 1, \\
&\text{or } s_{jim^*}(\alpha) + s_{kml^*}(\beta) + s_{kjn^*}(\chi) + s_{nil^*}(\delta) = 1 \text{ mod } 2.
\end{aligned} \tag{48}$$

When $N_{jim^*} < 1$ or $N_{kml^*} < 1$, the left-hand-side of eqn. (25) is always zero. Thus $F^{ijm,\alpha\beta,a}_{kln,\chi\delta,b} = 0$ when $N_{jim^*} < 1$ or $N_{kml^*} < 1$. When $N_{kjn^*} < 1$ or $N_{nil^*} < 1$, wave function on the right-hand-side of eqn. (25) is always zero. So we can choose $F^{ijm,\alpha\beta,a}_{kln,\chi\delta,b} = 0$ when $N_{kjn^*} < 1$ or $N_{nil^*} < 1$. Also, $F^{ijm,\alpha\beta,a}_{kln,\chi\delta,b}$ represents a pseudo-local bosonic operator which contains even numbers of fermionic operators. Therefore $F^{ijm,\alpha\beta,a}_{kln,\chi\delta,b}$ is non-zero only when $s_{jim^*}(\alpha) + s_{kml^*}(\beta) + s_{kjn^*}(\chi) + s_{nil^*}(\delta) = 0$ mod 2.

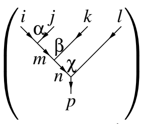
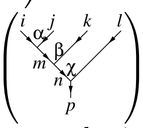
The F-move (30) can be viewed as a relation between wave functions on different v-graphs that are only differ by a local transformation. Since we can locally transform one v-graph to another v-graph through different paths, the F-move (30) must satisfy certain self consistent conditions. For example the v-graph  can be transformed to  through two different paths; one contains two steps of local transformations and another contains three steps of local transformations as described by eqn. (30). The two paths lead to the following relations between the wave functions:

$$\Psi_{\text{fix}}$$  $$\simeq \sum_{t\eta\varphi} \mathcal{F}^{ijm,\alpha\beta,\,\underline{\alpha\beta}}_{knt,\eta\varphi,\,\underline{\eta\varphi}} \Psi_{\text{fix}}$$  $$\simeq \sum_{t\eta\varphi;s\kappa\gamma} \mathcal{F}^{ijm,\alpha\beta,\,\underline{\alpha\beta}}_{knt,\eta\varphi,\,\underline{\eta\varphi}} \mathcal{F}^{itn,\varphi\chi,\,\underline{\varphi\chi}}_{lps,\kappa\gamma,\,\underline{\kappa\gamma}} \Psi_{\text{fix}}$$ 

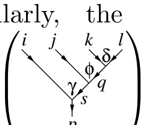
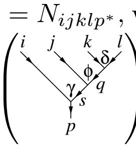
$$\simeq \sum_{t\eta\kappa;\varphi;s\kappa\gamma;q\delta\phi} \mathcal{F}^{ijm,\alpha\beta,\,\underline{\alpha\beta}}_{knt,\eta\varphi,\,\underline{\eta\varphi}} \mathcal{F}^{itn,\varphi\chi,\,\underline{\varphi\chi}}_{lps,\kappa\gamma,\,\underline{\kappa\gamma}} \mathcal{F}^{jkt,\eta\kappa,\,\underline{\eta\kappa}}_{lsq,\delta\phi,\,\underline{\delta\phi}} \Psi_{\text{fix}}$$  . (49)

$$\Psi_{\text{fix}}$$  $$\simeq \sum_{q\delta\epsilon} \mathcal{F}^{mkn,\beta\chi,\,\underline{\beta\chi}}_{lpq,\delta\epsilon,\,\underline{\delta\epsilon}} \Psi_{\text{fix}}$$  $$\simeq \sum_{q\delta\epsilon;s\phi\gamma} \mathcal{F}^{mkn,\beta\chi,\,\underline{\beta\chi}}_{lpq,\delta\epsilon,\,\underline{\delta\epsilon}} \mathcal{F}^{ijm,\alpha\epsilon,\,\underline{\alpha\epsilon}}_{qps,\phi\gamma,\,\underline{\phi\gamma}} \Psi_{\text{fix}}$$  , (50)

The consistence of the above two relations leads a condition on the F-tensor.

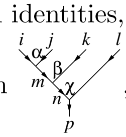
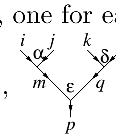
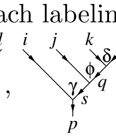
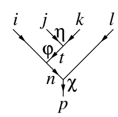
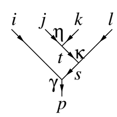
To obtain such a condition, let us fix i, j, k, l, p, and view ψ_{fix}  as a function of $\alpha, \beta, \chi, m, n$: $\phi(\alpha,\beta,\chi,m,n) = \psi_{\text{fix}}$ . As we vary indices on other part of graph, we obtain different wave functions $\phi(\alpha,\beta,\chi,m,n)$ which form a dimension D_{ijklp^*} space. In other words, D_{ijklp^*} is the support dimension of the state ψ_{fix} on the region α,β,χ,m,n with boundary state i,j,k,l,p (see the discussion in section V B). Since the number of choices of α,β,χ,m,n is $N_{ijklp^*} = \sum_{m,n} N_{jim^*}N_{kmn^*}N_{lnp^*}$, we have $D_{ijklp^*} \le N_{ijklp^*}$. Here we require a similar saturation condition as in (13):

$$N_{ijklp^*} = D_{ijklp^*} \tag{51}$$

Similarly, the number of choices of δ,ϕ,γ,q,s in ψ_{fix}  is also N_{ijklp^*}. Here we again assume $\tilde{D}_{ijklp^*} = N_{ijklp^*}$, where $\tilde{D}_{ijklp^*}$ is the support dimension of ψ_{fix}  on the region bounded by i,j,k,l,p.

So the two relations (50) and (49) can be viewed as two relations between a pair vectors in the two D_{ijklp^*} dimensional vector spaces. As we vary indices on other part of graph (still keep i,j,k,l,p fixed), each vector in the pair can span the full D_{ijklp^*} dimensional vector space. So the validity of the two relations (50) and (49) implies that

$$\sum_{t}\sum_{\eta=1}^{N_{kjt^*}}\sum_{\varphi=1}^{N_{tin^*}}\sum_{\kappa=1}^{N_{lts^*}} \mathcal{F}^{ijm,\alpha\beta,\,\underline{\alpha\beta}}_{knt,\eta\varphi,\,\underline{\eta\varphi}} \mathcal{F}^{itn,\varphi\chi,\,\underline{\varphi\chi}}_{lps,\kappa\gamma,\,\underline{\kappa\gamma}} \mathcal{F}^{jkt,\eta\kappa,\,\underline{\eta\kappa}}_{lsq,\delta\phi,\,\underline{\delta\phi}}$$
$$\simeq \sum_{\epsilon=1}^{N_{qmp^*}} \mathcal{F}^{mkn,\beta\chi,\,\underline{\beta\chi}}_{lpq,\delta\epsilon,\,\underline{\delta\epsilon}} \mathcal{F}^{ijm,\alpha\epsilon,\,\underline{\alpha\epsilon}}_{qps,\phi\gamma,\,\underline{\phi\gamma}}. \tag{52}$$

which is the fermionic generalization of the famous pentagon identity. The above expression actually contain many different pentagon identities, one for each labeling scheme of the vertices in , , , , and . Since the vertex labeling can be rearranged through eqn. (45), so we only need to consider one pentagon identity from a particular labeling scheme of the vertices: $\underline{\alpha} = \underline{\gamma} = \underline{\varphi} < \underline{\beta} = \underline{\epsilon} = \underline{\phi} = \underline{\eta} < \underline{\chi} = \underline{\delta} = \underline{\kappa}$. We obtain

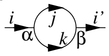
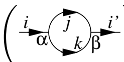
$$\sum_{t}\sum_{\eta=1}^{N_{kjt^*}}\sum_{\varphi=1}^{N_{tin^*}}\sum_{\kappa=1}^{N_{lts^*}} \mathcal{F}^{ijm,\alpha\beta,+}_{knt,\eta\varphi,-} \mathcal{F}^{itn,\varphi\chi,+}_{lps,\kappa\gamma,-} \mathcal{F}^{jkt,\eta\kappa,+}_{lsq,\delta\phi,-}$$
$$\simeq \sum_{\epsilon=1}^{N_{qmp^*}} \mathcal{F}^{mkn,\beta\chi,+}_{lpq,\delta\epsilon,-} \mathcal{F}^{ijm,\alpha\epsilon,+}_{qps,\phi\gamma,-}. \tag{53}$$

We can use the transformation (43) to change $\simeq$ to $=$ in the above equation and remove the Majorana numbers to rewrite the above as

$$\sum_{t}\sum_{\eta=1}^{N_{kjt^*}}\sum_{\varphi=1}^{N_{tin^*}}\sum_{\kappa=1}^{N_{lts^*}} F^{ijm,\alpha\beta,+}_{knt,\eta\varphi,-} F^{itn,\varphi\chi,+}_{lps,\kappa\gamma,-} F^{jkt,\eta\kappa,+}_{lsq,\delta\phi,-}$$
$$= (-)^{s_{jim^*}(\alpha)s_{lkq^*}(\delta)} \sum_{\epsilon=1}^{N_{qmp^*}} F^{mkn,\beta\chi,+}_{lpq,\delta\epsilon,-} F^{ijm,\alpha\epsilon,+}_{qps,\phi\gamma,-}. \tag{54}$$

The above fermionic pentagon identity (54) is a set of nonlinear equations satisfied by the rank-10 tensor $F^{ijm,\alpha\beta,+}_{kln,\chi\delta,-}$. The above consistency relations (44), (47), and (54) are equivalent to the requirement that the local unitary transformations described by eqn. (30) on different paths all commute with each other up to a total phase factor.

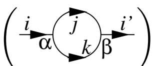
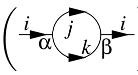
D. The second type of wave function renormalization

The second type of wave function renormalization does change the degrees of freedom and corresponds to a generalized local unitary transformation. One way to implement the second type renormalization is to reduce  to $\xrightarrow{i}$, so that we still have a trivalent graph. This requires that the support dimension $D_{ii'^*}$ of the fixed-point wave function ψ_{fix} $\Big($  $\Big)$ is given by

$$D_{ii'^*} = \delta_{ii'}. \tag{55}$$

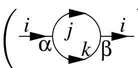
This implies that

$$\psi_{\text{fix}}\Big(\quad \Big) = \delta_{ii'}\psi_{\text{fix}}\Big(\quad \Big). \tag{56}$$

The second type renormalization can now be written as (since $D_{ii^*} = 1$)

$$\psi_{\text{fix}}^{\underline{\alpha}\,\underline{\beta}\,\underline{\eta}\cdots}\Big(\quad \Big) \simeq O^{jk,\alpha\beta}_{i,\,\underline{\alpha\beta}}\,\psi_{\text{fix}}^{\underline{\eta}\cdots}\Big(\xrightarrow{i}\Big). \tag{57}$$



where the ordering of the vertices is described by $\underline{\alpha}\,\underline{\beta}\,\underline{\eta}\ldots$. We will call such a wave function renormalization step a O-move. Here $O^{jk,\alpha\beta}_{i,a}$ satisfies

$$\sum_{k,j}\sum_{\alpha=1}^{N_{kii^*}}\sum_{\beta=1}^{N_{j^*jk^*}} O^{jk,\alpha\beta}_{i,a}(O^{jk,\alpha\beta}_{i,a})^* = 1 \tag{58}$$

and

$$O^{jk,\alpha\beta}_{i,a} = 0,\ \text{ if } N_{ik^*j^*} < 1,\ \text{ or } N_{i^*jk} < 1, \tag{59}$$
$$\text{or } s_{ik^*j^*}(\alpha) + s_{i^*jk}(\beta) = 1 \bmod 2.$$

The condition (58) ensures that the two wave functions on the two sides of eqn. (57) have the same normalization. We note that the number of choices for the four indices (j,k,α,β) in $O^{jk,\alpha\beta}_{i,\,\underline{\alpha\beta}}$ must be equal or greater than 1:

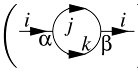
$$D_i = \sum_{jk} N_{ik^*j^*}N_{i^*jk} \geq 1. \tag{60}$$

In fact, we should have a stronger condition: the number of choices for the four indices (j,k,α,β) that correspond to bosonic states must be equal or greater than 1

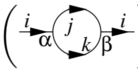
$$D_i = \sum_{jk} N^b_{ik^*j^*}N^b_{i^*jk} + N^f_{ik^*j^*}N^f_{i^*jk} \geq 1. \tag{61}$$

The wave functions in eqn. (57) is defined with respect to the ordering of the fermionic states described by $\underline{\alpha}\,\underline{\beta}\,\underline{\eta}\ldots$. Let us introduce

$$\Psi_{\text{fix}}\Big(\quad \Big) = [\theta_{\underline{\alpha}}^{s_{ik^*j^*}(\alpha)}\theta_{\underline{\beta}}^{s_{i^*jk}(\beta)}\theta_{\underline{\eta}}^{s(\eta)}\ldots]\times$$



$$\psi_{\text{fix}}^{\underline{\alpha}\,\underline{\beta}\,\underline{\eta}\cdots}\Big(\quad \Big),$$



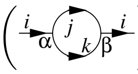
$$\Psi_{\text{fix}}\Big(\xrightarrow{i}\Big) = [\theta_{\underline{\eta}}^{s(\eta)}\ldots]\psi_{\text{fix}}^{\underline{\eta}\cdots}\Big(\xrightarrow{i}\Big) \tag{62}$$

and

$$\mathcal{O}^{jk,\alpha\beta}_{i,\,\underline{\alpha\beta}} = \theta_{\underline{\alpha}}^{s_{ik^*j^*}(\alpha)}\theta_{\underline{\beta}}^{s_{i^*jk}(\beta)}O^{jk,\alpha\beta}_{i,\,\underline{\alpha\beta}}. \tag{63}$$

We can rewrite (57) as

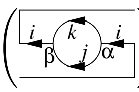
$$\Psi_{\text{fix}}\Big(\quad \Big) \simeq \mathcal{O}^{jk,\alpha\beta}_{i,\,\underline{\alpha\beta}}\Psi_{\text{fix}}\Big(\xrightarrow{i}\Big) \tag{64}$$



which is valid for any ordering of the fermionic states.

Notice that

$$\Psi_{\text{fix}}\Big(\quad \Big) \simeq \mathcal{O}^{k^*j^*,\beta\alpha}_{i^*,\,\underline{\beta\alpha}}\Psi_{\text{fix}}\Big(\xrightarrow{i}\Big). \tag{65}$$



Since eqn. (64) and eqn. (65) are really the same expression, we have

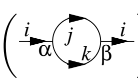
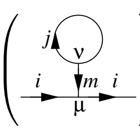
$$\mathcal{O}^{jk,\alpha\beta}_{i,a} \simeq \mathcal{O}^{k^*j^*,\beta\alpha}_{i^*,-a}. \tag{66}$$

Since the total phase of the wave function is unphysical, we can adjust the total phases of $\mathcal{O}^{jk,\alpha\beta}_{i,+}$ and $\mathcal{O}^{jk,\alpha\beta}_{i,-}$ independently. We can use this freedom to make eqn. (66) into $\mathcal{O}^{jk,\alpha\beta}_{i,a} = \mathcal{O}^{k^*j^*,\beta\alpha}_{i^*,-a}$ or

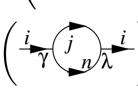
$$O^{jk,\alpha\beta}_{i,-} = (-)^{s_{ik^*j^*}(\alpha)s_{jki^*}(\beta)}O^{k^*j^*,\beta\alpha}_{i^*,+}. \tag{67}$$

We can find another relation between $\mathcal{O}^{jk,\alpha\beta}_{i,+}$ and $\mathcal{O}^{jk,\alpha\beta}_{i,-}$

$$\Psi_{\text{fix}}\Big(\quad \Big) \simeq \sum_{m\mu\nu}\mathcal{F}^{ij^*k,\alpha\beta,\,\underline{\alpha\beta}}_{jim,\nu\mu,\,\underline{\nu\mu}}\Psi_{\text{fix}}\Big(\quad \Big)$$

$$\simeq \sum_{m\mu\nu;n\gamma\lambda}\mathcal{F}^{ij^*k,\alpha\beta,\,\underline{\alpha\beta}}_{jim,\nu\mu,\,\underline{\nu\mu}}\mathcal{F}^{j^*jm,\nu\mu,\,\underline{\nu\mu}}_{i^*i^*n^*,\lambda\gamma,\,\underline{\lambda\gamma}}\Psi_{\text{fix}}\Big(\quad \Big), \tag{68}$$



which leads to

$$\mathcal{O}^{jk,\alpha\beta}_{i,a} \simeq \sum_{m\mu\nu;n\gamma\lambda}\mathcal{F}^{ij^*k,\alpha\beta,a}_{jim,\nu\mu,c}\mathcal{F}^{j^*jm,\nu\mu,c}_{i^*i^*n^*,\lambda\gamma,-b}\mathcal{O}^{jn,\gamma\lambda}_{i,b}. \tag{69}$$

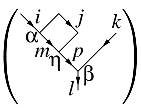
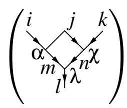
After removing the Majorana numbers, we find

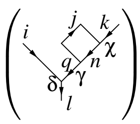
$$O^{jk,\alpha\beta}_{i,a} \simeq \sum_{m\mu\nu;n\gamma\lambda}(-)^{s_{in^*j^*}(\gamma)s_{jni^*}(\lambda)}F^{ij^*k,\alpha\beta,a}_{jim,\nu\mu,c}F^{j^*jm,\nu\mu,c}_{i^*i^*n^*,\lambda\gamma,-b}O^{jn,\gamma\lambda}_{i,b}. \tag{70}$$

Choosing $a=-$ and $b=c=+$ in the above, we find another relation between $\mathcal{O}^{jk,\alpha\beta}_{i,+}$ and $\mathcal{O}^{jk,\alpha\beta}_{i,-}$. Combining it with eqn. (67), we find

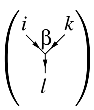
$$\mathrm{e}^{\mathrm{i}\,\theta_{O2}} O^{k^*j^*,\beta\alpha}_{i^*,+} = (-)^{s_{ik^*j^*}(\alpha) s_{jki^*}(\beta)} \times \sum_{m\mu\nu;n\gamma\lambda} (-)^{s_{in^*j^*}(\gamma) s_{jni^*}(\lambda)} F^{ij^*k,\alpha\beta,-}_{jim,\nu\mu,+} F^{j^*jm,\nu\mu,+}_{i^*i^*n^*,\lambda\gamma,-} O^{jn,\gamma\lambda}_{i,+}. \tag{71}$$

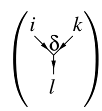
More conditions on $F^{ijm,\alpha\beta,a}_{kln,\chi\delta,b}$ and $O^{jk,\alpha\beta}_{i,a}$ can be obtained by noticing

Ψ_{fix}  $\simeq \sum_{n\chi\lambda} \mathcal{F}^{mjp,\eta\beta,\underline{\eta\beta}}_{kln,\chi\lambda,\underline{\chi\lambda}} \Psi_{\text{fix}}$ 

$\simeq \sum_{n\chi\lambda;q\gamma\delta} \mathcal{F}^{mjp,\eta\beta,\underline{\eta\beta}}_{kln,\chi\lambda,\underline{\chi\lambda}} \mathcal{F}^{ij^*m,\alpha\lambda,\underline{\alpha\lambda}}_{nlq,\gamma\delta,\underline{\gamma\delta}} \Psi_{\text{fix}}$  , (72)

which implies that

$\mathcal{O}^{jm,\alpha\eta}_{i,\underline{\alpha\eta}} \delta_{ip} \Psi_{\text{fix}}$  (73)

$\simeq \sum_{n\chi\lambda;\gamma\delta} \mathcal{F}^{mjp,\eta\beta,\underline{\eta\beta}}_{kln,\chi\lambda,\underline{\chi\lambda}} \mathcal{F}^{ij^*m,\alpha\lambda,\underline{\alpha\lambda}}_{nlk,\gamma\delta,\underline{\gamma\delta}} \mathcal{O}^{jn^*,\gamma\chi}_{k^*,\underline{\gamma\chi}} \Psi_{\text{fix}}$  .

We find that

$$\begin{aligned} &\mathcal{O}^{jm,\alpha\eta}_{i,\underline{\alpha\eta}} \delta_{ip}\delta_{\beta\delta} \theta_{\underline{\beta}}^{s_{kil^*}(\beta)} \\ &\simeq \sum_{n\chi\lambda\gamma} \mathcal{F}^{mjp,\eta\beta,\underline{\eta\beta}}_{kln,\chi\lambda,\underline{\chi\lambda}} \mathcal{F}^{ij^*m,\alpha\lambda,\underline{\alpha\lambda}}_{nlk,\gamma\delta,\underline{\gamma\delta}} \mathcal{O}^{jn^*,\gamma\chi}_{k^*,\underline{\gamma\chi}} \theta_{\underline{\delta}}^{s_{kil^*}(\delta)} \\ &\text{for all } k,i,l \text{ satisfying } N_{kil^*}>0. \end{aligned} \tag{74}$$

If we choose a particular labeling of the vertices

$$\alpha=\delta<\eta=\lambda=\gamma<\beta=\chi, \tag{75}$$

we reduce the above expression to

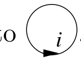
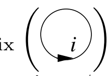
$$\begin{aligned} &\mathcal{O}^{jm,\alpha\eta}_{i,+} \delta_{ip}\delta_{\beta\delta} \theta_{\underline{\beta}}^{s_{kil^*}(\beta)} \\ &\simeq \sum_{n\chi\lambda\gamma} \mathcal{F}^{mjp,\eta\beta,+}_{kln,\chi\lambda,-} \mathcal{F}^{ij^*m,\alpha\lambda,+}_{nlk,\gamma\delta,-} \mathcal{O}^{jn^*,\gamma\chi}_{k^*,+} \theta_{\underline{\delta}}^{s_{kil^*}(\delta)} \\ &\text{for all } k,i,l \text{ satisfying } N_{kil^*}>0, \end{aligned} \tag{76}$$

or

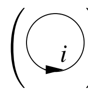
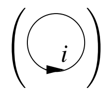
$$\begin{aligned} \mathrm{e}^{\mathrm{i}\,\theta_{O1}} O^{jm,\alpha\eta}_{i,+} \delta_{ip}\delta_{\beta\delta} = &\sum_{n\chi\lambda\gamma} (-)^{s_{kjn^*}(\chi)[s_{kjn^*}(\chi)+s_{jmp^*}(\eta)]} \times \\ &F^{mjp,\eta\beta,+}_{kln,\chi\lambda,-} F^{ij^*m,\alpha\lambda,+}_{nlk,\gamma\delta,-} O^{jn^*,\gamma\chi}_{k^*,+} \\ &\text{for all } k,i,l \text{ satisfying } N_{kil^*}>0. \end{aligned} \tag{77}$$

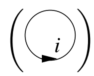
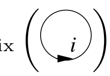
E. The fixed-point wave functions from the fixed-point gfLU transformations

In the last section, we discussed the conditions that a fixed-point gfLU transformation $(F^{ijm,\alpha\beta,a}_{kln,\gamma\lambda,b}, O^{jk,\alpha\beta}_{i,a})$ must satisfy. After finding a fixed-point gfLU transformation $(F^{ijm,\alpha\beta,a}_{kln,\gamma\lambda,b}, O^{jk,\alpha\beta}_{i,a})$, in this section, we are going to discuss how to calculate the corresponding fixed-point wave function ψ_{fix} from the solved fixed-point gfLU transformation $(F^{ijm,\alpha\beta,a}_{kln,\gamma\lambda,b}, O^{jk,\alpha\beta}_{i,a})$.

First we note that, using the two types of the wave function renormalization introduced above, we can reduce any graph to . So, once we know ψ_{fix} , we can reconstruct the full fixed-point wave function ψ_{fix} on any connected graph.

Let us assume that

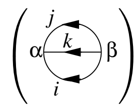
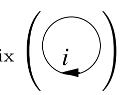
Ψ_{fix}  $= \psi_{\text{fix}}$  $= A^i$. (78)

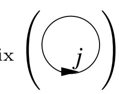
Note that Ψ_{fix}  contains no Majorana numbers and thus is equal to ψ_{fix}  . Here A^i satisfy

$$\sum_i A^i (A^i)^* = 1, \tag{79}$$

which is simply the normalization condition of the wave function.

To find conditions on A^i, we note that

Ψ_{fix}  $\simeq \mathcal{O}^{j^*k^*,\alpha\beta}_{i,\underline{\alpha\beta}} \Psi_{\text{fix}}$ 

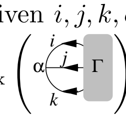
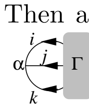
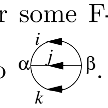
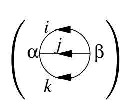
$\simeq \mathcal{O}^{k^*i^*,\alpha\beta}_{j,\underline{\alpha\beta}} \Psi_{\text{fix}}$  . (80)

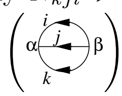
This allows us to show

$$\mathcal{O}^{j^*k^*,\alpha\beta}_{i,a} A^{i^*} = \mathcal{O}^{k^*i^*,\alpha\beta}_{j,a} A^j, \tag{81}$$

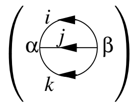
or

$$O^{j^*k^*,\alpha\beta}_{i,a} A^{i^*} = \mathrm{e}^{\mathrm{i}\,\theta^a_A} O^{k^*i^*,\alpha\beta}_{j,a} A^j. \tag{82}$$

Also, for any given i,j,k,α that satisfy $N_{kji}>0$, the wave function ψ_{fix}  must be non-zero for some Γ, where Γ represents indices on other part of the graph. Then after some F-moves and O-moves, we can reduce  to . So, for any given i,j,k,α that satisfy $N_{kji}>0$, ψ_{fix}  is non-zero for some β.

Since such a statement is true for any choices of basis on the vertex α, we find that for any given i, j, k, α that satisfy $N_{kji} > 0$ and for any non-zero vector v_α, $\sum_\alpha v_\alpha \psi_{\text{fix}}$  is non-zero for some β. This means that the matrix $M_{kji,a}$ given by

$$(M_{kji,a})_{\alpha\beta} \equiv \psi_{\text{fix}} \quad = O^{j^*k^*,\alpha\beta}_{i,a} A^{i^*} \tag{83}$$



is invertible:

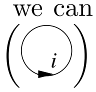
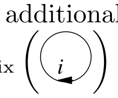
$$\det(M_{kji,a}) \neq 0. \tag{84}$$

The above also implies that

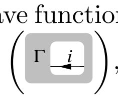
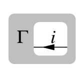
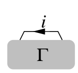
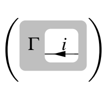
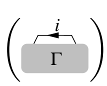
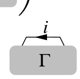
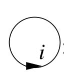
$$N_{kji} = N_{i^*j^*k^*}. \tag{85}$$

Such a condition ensures that eqn. (60) is always valid. So we will drop eqn. (60). Due to the mod 2 conservation of fermion numbers, eqn. (59) also allows us to show that

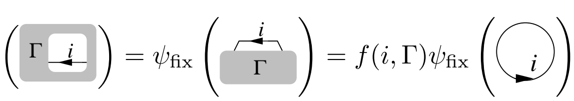
$$N^f_{kji} = N^f_{i^*j^*k^*}. \tag{86}$$

If we further assume that the wave function is well defined on a sphere, we can have additional conditions. One of them is ψ_{fix}  $= \psi_{\text{fix}}$  which leads to

$$A^i = A^{i^*}. \tag{87}$$

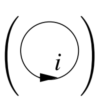
The second one is $A^i \neq 0$. To show this, let us first consider the fixed-point wave function where the index on a link is i: $\psi_{\text{fix}}(i, \Gamma) = \psi_{\text{fix}}$ , where Γ are indices on other part of graph. We note that the v-graph  can be deformed into the v-graph  on a sphere. Thus ψ_{fix}  $= \psi_{\text{fix}}$ . Using the F-moves and the O-moves, we can reduce  to :

$$\psi_{\text{fix}} \quad = \psi_{\text{fix}} \quad = f(i,\Gamma)\psi_{\text{fix}} \tag{88}$$

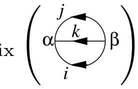
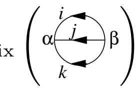


We see that

$$\psi_{\text{fix}} \quad = A^i \neq 0 \tag{89}$$



for all i. Otherwise, any wave function with i-link will be zero.

The third condition comes from Ψ_{fix}  $=$ Ψ_{fix} , which implies $\mathcal{O}^{j^*k^*,\alpha\beta}_{i,a} A^{i^*} = \mathcal{O}^{i^*j^*,\alpha\beta}_{k,a} A^{k^*}$ or

$$O^{j^*k^*,\alpha\beta}_{i,a} A^{i^*} = O^{i^*j^*,\alpha\beta}_{k,a} A^{k^*}, \tag{90}$$

In fact the above relation and eqn. (79) allow us to show $\theta^a_A = 0$ in eqn. (82). When $\theta^a_A = 0$, above relation can be derived from eqn. (79) and eqn. (82) and thus can be dropped. The conditions (79, 82, 84, 87, 89) allow us to determine A^i.

F. Summary of fixed-point gfLU transformations

To summarize, the conditions (16, 18, 61, 85, 86, 31, 42, 47, 48, 54, 58, 59, 67, 71, 77, 79, 82, 84, 87, 89) form a set of non-linear equations whose variables are N_{ijk}, N^f_{ijk}, $F^{ijm,\alpha\beta,a}_{kln,\gamma\lambda,b}$, $O^{jk,\alpha\beta}_{i,a}$, A^i, and the seven phases $(e^{i\theta^{ab}_F}|_{a=\pm,b=\pm}, e^{i\theta_{O1}}, e^{i\theta_{O2}})$. Let us collect those conditions and list them below

$$\bullet \sum_{m=0}^{N} N_{jim^*} N_{kml^*} = \sum_{n=0}^{N} N_{kjn^*} N_{l^*ni},$$

$$\bullet \sum_{m=0}^{N} (N^b_{jim^*} N^f_{kml^*} + N^f_{jim^*} N^b_{kml^*}) = \sum_{n=0}^{N} (N^b_{kjn^*} N^f_{l^*ni} + N^f_{kjn^*} N^b_{l^*ni}),$$

$$\bullet \sum_{jk} (N^b_{ik^*j^*} N^b_{i^*jk} + N^f_{ik^*j^*} N^f_{i^*jk}) \geq 1,$$

$$\bullet\ N_{kji} = N_{i^*j^*k^*}, \quad \bullet\ N^f_{kji} = N^f_{i^*j^*k^*}. \tag{91}$$

$$\bullet \sum_{n\chi\delta} F^{ijm',\alpha'\beta',\underline{\alpha\beta}}_{kln,\chi\delta,\underline{\chi\delta}} (F^{ijm,\alpha\beta,\underline{\alpha\beta}}_{kln,\chi\delta,\underline{\chi\delta}})^* = \delta_{m\alpha\beta,m'\alpha'\beta'},$$

$$\bullet\ (F^{ijm,\alpha\beta,a}_{kln,\chi\delta,b})^* = (-)^{s_{jim^*}(\alpha)s_{kml^*}(\beta)} F^{jkn,\chi\delta,b}_{l^*i^*m^*,\beta\alpha,-a},$$

$$\bullet \sum_{n\chi\delta} F^{ijm,\alpha\beta,a}_{kln,\chi\delta,+} (F^{ijm',\alpha'\beta',b}_{kln,\chi\delta,+})^* = e^{i\theta^{ab}_F} \sum_{n\chi\delta} F^{ijm,\alpha\beta,a}_{kln,\chi\delta,-} (F^{ijm',\alpha'\beta',b}_{kln,\chi\delta,-})^*,$$

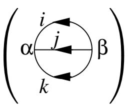
- $F^{ijm,\alpha\beta,a}_{kln,\chi\delta,b} = 0$ when $N_{jim^*} < 1$ or $N_{kml^*} < 1$ or $N_{kjn^*} < 1$ or $N_{nil^*} < 1$, or $s_{jim^*}(\alpha) + s_{kml^*}(\beta) + s_{kjn^*}(\chi) + s_{nil^*}(\delta) =$ odd,

$$\bullet \sum_t \sum_{\eta=1}^{N_{kjt^*}} \sum_{\varphi=1}^{N_{tin^*}} \sum_{\kappa=1}^{N_{lts^*}} F^{ijm,\alpha\beta,+}_{knt,\eta\varphi,-} F^{itn,\varphi\chi,+}_{lps,\kappa\gamma,-} F^{jkt,\eta\kappa,+}_{lsq,\delta\phi,-} = (-)^{s_{jim^*}(\alpha)s_{lkq^*}(\delta)} \sum_{\epsilon=1}^{N_{qmp^*}} F^{mkn,\beta\chi,+}_{lpq,\delta\epsilon,-} F^{ijm,\alpha\epsilon,+}_{qps,\phi\gamma,-}. \tag{92}$$

- $\sum_{k,j}\sum_{\alpha=1}^{N_{kii^*}}\sum_{\beta=1}^{N_{j^*jk^*}} O_{i,a}^{jk,\alpha\beta}(O_{i,a}^{jk,\alpha\beta})^* = 1,$
- $O_{i,a}^{jk,\alpha\beta} = 0,$ if $N_{ik^*j^*} < 1$, or $N_{i^*jk} < 1$, or $s_{ik^*j^*}(\alpha) + s_{i^*jk}(\beta) =$ odd,
- $O_{i,-}^{jk,\alpha\beta} = (-)^{s_{ik^*j^*}(\alpha)s_{jki^*}(\beta)} O_{i^*,+}^{k^*j^*,\beta\alpha},$
- $\mathrm{e}^{\mathrm{i}\,\theta_{O2}} O_{i^*,+}^{k^*j^*,\beta\alpha} = (-)^{s_{ik^*j^*}(\alpha)s_{jki^*}(\beta)} \times \sum_{m\mu\nu;n\gamma\lambda} (-)^{s_{in^*j^*}(\gamma)s_{jni^*}(\lambda)} F_{jim,\nu\mu,+}^{ij^*k,\alpha\beta,-} F_{i^*i^*n^*,\lambda\gamma,-}^{j^*jm,\nu\mu,+} O_{i,+}^{jn,\gamma\lambda},$
- $\mathrm{e}^{\mathrm{i}\,\theta_{O1}} O_{i,+}^{jm,\alpha\eta}\delta_{ip}\delta_{\beta\delta} = \sum_{n\chi\lambda\gamma} (-)^{s_{kjn^*}(\chi)[s_{kjn^*}(\chi)+s_{jmp^*}(\eta)]} \times F_{kln,\chi\lambda,-}^{mjp,\eta\beta,+} F_{nlk,\gamma\delta,-}^{ij^*m,\alpha\lambda,+} O_{k^*,+}^{jn^*,\gamma\chi}$

for all k, i, l satisfying $N_{kil^*} > 0$. (93)

- $\sum_i A^i (A^i)^* = 1,$ • $A^i = A^{i^*} \neq 0,$
- $O_{i,a}^{j^*k^*,\alpha\beta} A^{i^*} = O_{j,a}^{k^*i^*,\alpha\beta} A^j,$
- $\det(M_{kji,\underline{\alpha\beta}}) \equiv \det\left[\psi_{\text{fix}}\left(\quad\right)\right] \neq 0.$ (94)



Finding N_{ijk}, N_{ijk}^f, $F_{kln,\gamma\lambda,b}^{ijm,\alpha\beta,a}$, $O_{i,a}^{jk,\alpha\beta}$, and A^i that satisfy such a set of non-linear equations corresponds to finding a fixed-point gfLU transformation that has a non-trivial fixed-point wave function. So the solutions $(N_{ijk}, N_{ijk}^f, F_{kln,\gamma\lambda,b}^{ijm,\alpha\beta,a}, O_{i,a}^{jk,\alpha\beta}, A^i)$ give us a characterization of fermionic topological orders (and bosonic topological orders as a special case where $N_{ijk}^f = 0$).

VI. GENERALIZED TURAEV-VIRO INVARIANCE AND ITS GRAPHIC REPRESENTATION

As we know, the string-net model has a very deep relationship with the Turaev-Viro invariance[36] for closed triangulated 3-manifolds.[37,38] In this section, we will show how such a states sum invariance is generalized to its fermionic version.

A. The amplitude of tetrahedra graph and fermionic symmetric $\mathcal{G}$ symbol

Let us define the amplitude of the tetrahedra graph in Fig. 4 as $\mathcal{G}_{kln,\chi\delta}^{ijm,\alpha\beta}$. We call such a Majorana number valued tetrahedra amplitude as the fermionic symmetric $\mathcal{G}$ symbol if it is well defined on a sphere and has the twelve tetrahedral symmetry. Notice here we assume the $\mathcal{G}$ symbol is independent of the labeling scheme $\underline{\alpha\beta\chi\delta}$. Such an assumption is a consequence of the tetrahedral symmetry and will be proved later.

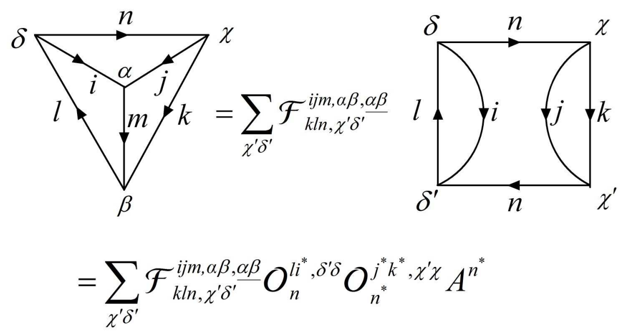


FIG. 4. We define the $\mathcal{G}$ symbol as the amplitude of the tetrahedra graph, which can be expressed in terms of $\mathcal{F}$, $\mathcal{O}$ and A^i.

In terms of Majorana number, we can express the amplitude of the tetrahedra graph as:

$$\begin{aligned}
\mathcal{G}_{kln,\chi\delta}^{ijm,\alpha\beta,\underline{\alpha\beta}} &= \sum_{\chi'\delta'} \mathcal{F}_{kln,\chi'\delta',\underline{\chi'\delta'}}^{ijm,\alpha\beta,\underline{\alpha\beta}} \mathcal{O}_{n,\underline{\delta'\delta}}^{li^*,\delta'\delta} \mathcal{O}_{n^*,\underline{\chi'\chi}}^{j^*k^*,\chi'\chi} A^{n^*} \\
&= \theta_{\underline{\alpha}}^{s_{jim^*}(\alpha)} \theta_{\underline{\beta}}^{s_{kml^*}(\beta)} \theta_{\underline{\delta}}^{s_{nil^*}(\delta)} \theta_{\underline{\chi}}^{s_{kjn^*}(\chi)} \times \\
&\quad \sum_{\chi'\delta'} F_{kln,\chi'\delta',\underline{\chi'\delta'}}^{ijm,\alpha\beta,\underline{\alpha\beta}} O_{n,\underline{\delta'\delta}}^{li^*,\delta'\delta} O_{n^*,\underline{\chi'\chi}}^{j^*k^*,\chi'\chi} A^{n^*} \\
&= \theta_{\underline{\alpha}}^{s_{jim^*}(\alpha)} \theta_{\underline{\beta}}^{s_{kml^*}(\beta)} \theta_{\underline{\delta}}^{s_{nil^*}(\delta)} \theta_{\underline{\chi}}^{s_{kjn^*}(\chi)} G_{kln,\chi\delta}^{ijm,\alpha\beta,\underline{\alpha\beta}},
\end{aligned} \tag{95}$$

with

$$G_{kln,\chi\delta}^{ijm,\alpha\beta,\underline{\alpha\beta}} = \sum_{\chi'\delta'} F_{kln,\chi'\delta',\underline{\chi'\delta'}}^{ijm,\alpha\beta,\underline{\alpha\beta}} O_{n,\underline{\delta'\delta}}^{li^*,\delta'\delta} O_{n^*,\underline{\chi'\chi}}^{j^*k^*,\chi'\chi} A^{n^*} \tag{96}$$

Notice the above expression of $\mathcal{G}$ makes it only depend on the labeling scheme of $\underline{\alpha\beta}$ but not $\underline{\chi'\delta'}, \underline{\chi'\chi'}, \underline{\delta'\delta}$ due to the relabeling equations on $\mathcal{F}$ and $\mathcal{O}$. For example, no matter what value $\underline{\chi\delta}$ takes, we can always chose $\underline{\chi'\chi'} = +$, $\underline{\delta'\delta} = +$ and freely chose the value $\underline{\chi'\delta'}$.

If the wavefunction is well defined on a sphere, then the $\mathcal{G}$ symbol must satisfy the following twelve tetrahedral symmetry as shown in Fig. 5 (a),(b) and (c):

$$\mathcal{G}_{kln,\chi\delta}^{ijm,\alpha\beta,\underline{\alpha\beta}} = \mathcal{G}_{ij^*n^*,\delta\chi}^{kl^*m^*,\beta\alpha,\underline{\beta\alpha}} = \mathcal{G}_{nkl,\beta\delta}^{m^*ij^*,\alpha\chi,\underline{\alpha\chi}}. \tag{97}$$

The first equality is nothing but the consequence of $\mathcal{F}_{kln,\chi\delta,\underline{\chi\delta}}^{ijm,\alpha\beta,\underline{\alpha\beta}} = \mathcal{F}_{ij^*n^*,\delta\chi,\underline{\delta\chi}}^{kl^*m^*,\alpha\beta,\underline{\alpha\beta}}$, however, the second equality is a new condition on $\mathcal{F}$ and $\mathcal{O}$.

The relation $\mathcal{G}_{kln,\chi\delta}^{ijm,\alpha\beta,\underline{\alpha\beta}} = \mathcal{G}_{nkl,\beta\delta}^{m^*ij^*,\alpha\chi,\underline{\alpha\chi}}$ implies that $\mathcal{G}_{kln,\chi\delta}^{ijm,\alpha\beta,\underline{\alpha\beta}}$ is independent on the labeling scheme $\underline{\alpha\beta}$. This is because we can choose $\underline{\alpha\beta}$ and $\underline{\alpha\chi}$ equal to $\pm$ independently. Thus we can finally get rid of the labeling $\underline{\alpha\beta}$and rewrite the $\mathcal{G}$ symbol as:

$$\mathcal{G}_{kln,\chi\delta}^{ijm,\alpha\beta} = \theta_{\underline{\alpha}}^{s_{jim^*}(\alpha)} \theta_{\underline{\beta}}^{s_{kml^*}(\beta)} \theta_{\underline{\delta}}^{s_{nil^*}(\delta)} \theta_{\underline{\chi}}^{s_{kjn^*}(\chi)} G_{kln,\chi\delta}^{ijm,\alpha\beta} \tag{98}$$

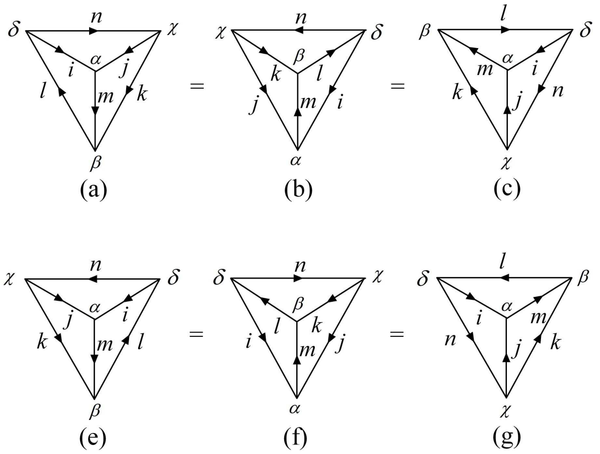


FIG. 5. (a) The tetrahedron represents G symbol. (b) The tetrahedron obtained by rotating (a) around the axis connecting the centers of the link m and n by $180°$. (c) The tetrahedron obtained by rotating (a) around the center by $180°$.(e),(f),(g) are the mirror reflection of (a),(b),(c).

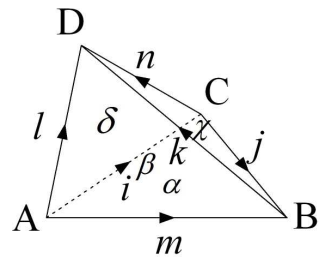


FIG. 6. The graphic representation of the G symbol $G^{ijm,\alpha\beta}_{kln,\chi\delta}$

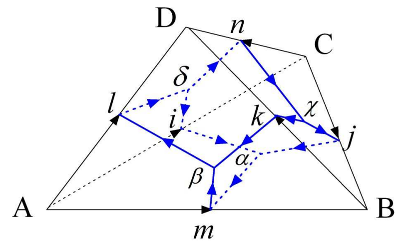


FIG. 7. (Color online) The tetrahedra graphic representation of the G symbol is actually a due representation of the original tetrahedra graph.

The tetrahedral relation can also be reexpressed as:

$$\mathcal{G}^{ijm,\alpha\beta}_{kln,\chi\delta} = \mathcal{G}^{kl^*m^*,\beta\alpha}_{ij^*n^*,\delta\chi} = \mathcal{G}^{m^*ij^*,\alpha\chi}_{nkl,\beta\delta}. \tag{99}$$

The amplitudes of the corresponding mirror reflection of the above tetrahedron graphs (see Fig. 5 (e), (f), (g)) take different values but also have the twelve tetrahedral symmetries.

$$\mathcal{G}^{jim,\alpha\beta}_{l^*k^*n^*,\delta\chi} = \mathcal{G}^{l^*km^*,\beta\alpha}_{ji^*n,\chi\delta} = \mathcal{G}^{m^*ij^*,\alpha\chi}_{n^*k^*l^*,\beta\delta}. \tag{100}$$

B. Graphic representation for $\mathcal{G}$ symbol

1. *Graphic representation in the bosonic case*

To explain the graphic representation of the Majorana number valued $\mathcal{G}$ symbol, let us first consider a simple case that all the vertex indices $\alpha\beta\chi\delta$ are bosonic ($s_{jim^*}(\alpha) = s_{kml^*}(\beta) = s_{nil^*}(\delta) = s_{kjn^*}(\chi) = 0$). In this case, the $\mathcal{G}$ symbol is equivalent to the G symbol.

We can use a tetrahedra to represent the G symbol. the ten indices of the G symbol can be put on the six oriented links and four faces. The detail rule is defined as following:

(a) We choose an arbitrary surface of the tetrahedra, put the three upper indices ijm of the G symbol on the edge of this surface and the lower indices kln on the rest three links of the tetrahedra. Notice for each upper index, the lower index should always be put on the opposite link on the tetrahedra accordingly.

(b) We introduce a self-consistent rule to define the orientations of the links. We may put ijm onto the edge of the surface ABC with clockwise ordering and the orientations of i, j links are clockwise while the orientation of m link is anticlockwise(looking from the top vertex D of the tetrahedra). The remaining three links all point to the top vertex D. Because three link indices in the G symbol uniquely determine a vertex index(e.g., the ijm link indices determine the α vertex labeled as $\underline{\alpha}$), it allows us to put the vertex indices of the G symbol onto the faces of the tetrahedra accordingly. Here we use the convention that the two upper indices $\alpha\beta$ correspond to faces ijm and klm, and the two lower indices $\chi\delta$ correspond to faces kjn and iln.

(c) If the orientation of a link i is opposite to the convention we choose in (b), then we simply replace it with its dual i^*.

Essentially, the above graphic representation is nothing but the dual representation of the original tetrahedra graph, as shown in Fig. 7. The twelve tetrahedra symmetries of the G symbol are also automatically implemented in such a dual representation.

2. *Graphic representation for the full $\mathcal{G}$ symbol*

Since the Majorana numbers only appear for those vertex indices, it is straightforward to generalize the above graphic representation into the fermionic case. We only need to put Majorana numbers $\theta_{\underline{\alpha}}$ onto each surface of the tetrahedra to represent those fermionic states and have:

$$\mathcal{G}^{ijm,\alpha\beta}_{kln,\chi\delta} = \theta_{\underline{\alpha}}^{s_{jim^*}(\alpha)}\theta_{\underline{\beta}}^{s_{kml^*}(\beta)}\theta_{\underline{\delta}}^{s_{nil^*}(\delta)}\theta_{\underline{\chi}}^{s_{kjn^*}(\chi)}G^{ijm,\alpha\beta}_{kln,\chi\delta}. \tag{101}$$

Notice here we order the Majorana number in the same convention as in the $\mathcal{F}$ symbol.

C. The $2 \to 3/3 \to 2$ move for the $\mathcal{G}$ symbol

1. The orthogonality condition

To derive the $2 \to 3/3 \to 2$ move for the $\mathcal{G}$ symbol from the self-consistent equations of $\mathcal{F}$ and $\mathcal{O}$, we need to assume the vertex indices in $\mathcal{O}$ satisfy the following orthogonality condition:

$$\mathcal{O}^{jk,\alpha\beta}_{i,\underline{\alpha\beta}} = \theta_{\underline{\alpha}}^{s_{ik^*j^*}(\alpha)} \theta_{\underline{\beta}}^{s_{i^*jk}(\beta)} O^{jk}_{i,\underline{\alpha\beta}} \delta_{\alpha\beta} \tag{102}$$

This condition is automatically satisfied in the (bosonic) string-net theory and we believe it is also true for a large class of fermionic topologically ordered states.

Under such an orthogonality condition, we can simplify the expression of G as:

$$G^{ijm,\alpha\beta}_{kln,\chi\delta} = F^{ijm,\alpha\beta,\underline{\alpha\beta}}_{kln,\chi\delta,\underline{\chi'\delta'}} O^{li^*}_{n,\underline{\delta'\delta}} O^{j^*k^*}_{n^*,\underline{\chi'\chi}} A^{n^*}. \tag{103}$$

Because $G^{ijm,\alpha\beta}_{kln,\chi\delta}$ is independent on the labeling schemes $\underline{\alpha\beta}$, $\underline{\chi'\delta'}$, $\underline{\delta'\delta}$, $\underline{\chi'\chi}$, we can show F and O tensors are actually also independent on those labeling schemes. Hence, we can omit the labeling indices for the Majorana number valued unitary transformation $\mathcal{F}$ and $\mathcal{O}$:

$$\begin{aligned}\mathcal{F}^{ijm,\alpha\beta}_{kln,\chi\delta} &= \theta_{\underline{\alpha}}^{s_{jim^*}(\alpha)} \theta_{\underline{\beta}}^{s_{kml^*}(\beta)} \theta_{\underline{\delta}}^{s_{nil^*}(\delta)} \theta_{\underline{\chi}}^{s_{kjn^*}(\chi)} F^{ijm,\alpha\beta}_{kln,\chi\delta} \\ \mathcal{O}^{jk,\alpha\beta}_{i,\underline{\alpha\beta}} &= \theta_{\underline{\alpha}}^{s_{ik^*j^*}(\alpha)} \theta_{\underline{\beta}}^{s_{i^*jk}(\beta)} O^{jk}_{i} \delta_{\alpha\beta}\end{aligned} \tag{104}$$

Finally, we can omit all the labeling indices and simplify the expression of G as:

$$G^{ijm,\alpha\beta}_{kln,\chi\delta} = F^{ijm,\alpha\beta}_{kln,\chi\delta} O^{li^*}_{n} O^{j^*k^*}_{n^*} A^{n^*}. \tag{105}$$

2. Graphic representation for the $\mathbb{\Theta}$ symbol

To derive the $2 \to 3/3 \to 2$ move for the $\mathcal{G}$ symbol, we also would like to introduce the graphic representation for the $\mathbb{\Theta}$ symbol, which is defined as the amplitude of the Θ graph.

In the dual representation, the Θ graph can be represented as a triangle, see in Fig. 8.

$$\mathbb{\Theta}^{kji,\alpha\beta}_{\underline{\alpha\beta}} = \mathcal{O}^{jk,\alpha\beta}_{i^*,\underline{\alpha\beta}} A^i = \theta_{\underline{\alpha}}^{s_{i^*k^*j^*}(\alpha)} \theta_{\underline{\beta}}^{s_{ijk}(\beta)} O^{jk,\alpha\beta}_{i^*,\underline{\alpha\beta}} A^i. \tag{106}$$

We put the kji indices on the links of the triangle and the indices $\alpha\beta$ on the front surface and opposite surface of the triangle, then we use the right hand rule to determine the orientations of the links and the ordering of the $\alpha\beta$ indices. If the Θ graph is well defined on a sphere, we can derive the cyclic symmetry:

$$\mathbb{\Theta}^{kji,\alpha\beta}_{\underline{\alpha\beta}} = \mathbb{\Theta}^{ikj,\alpha\beta}_{\underline{\alpha\beta}} = \mathbb{\Theta}^{jik,\alpha\beta}_{\underline{\alpha\beta}} \tag{107}$$

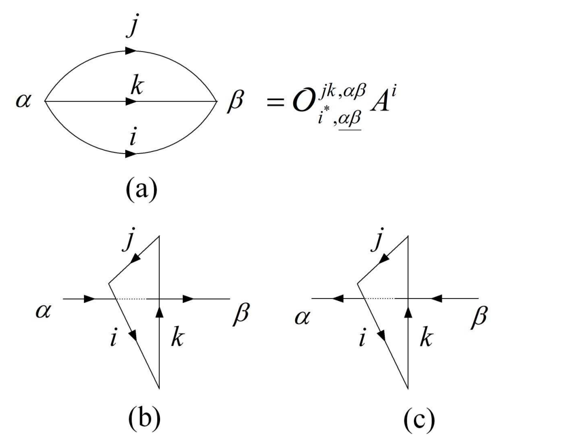


FIG. 8. The graphic representation of the amplitude of Θ graph and its inverse.

See in Fig. 8 (b), using the right hand rule, the graphic representation indicates the cyclic symmetry as well as the 180 degree rotation symmetry:

$$\mathbb{\Theta}^{kji,\alpha\beta}_{\underline{\alpha\beta}} = \mathbb{\Theta}^{j^*k^*i^*,\beta\alpha}_{\underline{\beta\alpha}}, \tag{108}$$

which is automatically satisfied from the condition $\mathcal{O}^{jk,\alpha\beta}_{i^*,\underline{\alpha\beta}} = \mathcal{O}^{k^*j^*,\beta\alpha}_{i,\underline{\beta\alpha}}$ and $A^i = A^{i^*}$.

Similarly, we can use the left hand rule to define the inverse of $\mathbb{\Theta}$(for those nonzero component) as:

$$\left(\mathbb{\Theta}^{kji,\alpha\beta}_{\underline{\alpha\beta}}\right)^{-1} = \theta_{\underline{\beta}}^{s_{ijk}(\beta)} \theta_{\underline{\alpha}}^{s_{i^*k^*j^*}(\alpha)} \left(O^{jk,\alpha\beta}_{i^*,\underline{\alpha\beta}} A^i\right)^{-1} \delta_{ijk}, \tag{109}$$

where $\delta_{ijk} = 1$ for $N_{ijk} > 0$ and $\delta_{ijk} = 0$ for $N_{ijk} = 0$.

If $\mathcal{O}$ symbol satisfy the orthogonality condition, we can further simplify the above expression for Θ graph as:

$$\mathbb{\Theta}^{kji,\alpha\beta} = \mathcal{O}^{jk,\alpha\beta}_{i^*} A^i = \theta_{\underline{\alpha}}^{s_{i^*k^*j^*}(\alpha)} \theta_{\underline{\beta}}^{s_{ijk}(\beta)} O^{jk}_{i^*} A^i \delta_{\alpha\beta}, \tag{110}$$

where the amplitude of Θ graph is also independent on the labeling scheme. The inverse of $\mathbb{\Theta}$ can also be simplified accordingly:

$$\left(\mathbb{\Theta}^{kji,\alpha\beta}\right)^{-1} = \theta_{\underline{\beta}}^{s_{ijk}(\beta)} \theta_{\underline{\alpha}}^{s_{i^*k^*j^*}(\alpha)} \left(O^{jk}_{i^*} A^i\right)^{-1} \delta_{\alpha\beta}\delta_{ijk}. \tag{111}$$

In this case, the cyclic condition (107) on Θ symbol will further imply:

$$O^{jk}_{i^*} A^i = O^{ki}_{j^*} A^j = O^{ij}_{k^*} A^k. \tag{112}$$

3. The $2 \to 3/3 \to 2$ move

Let us first consider a simple case that all the vertex indices $\alpha\beta\chi\delta$ of $\mathcal{G}$ symbol are bosonic($s_{jim^*}(\alpha) =$

$s_{kml^*}(\beta) = s_{nil^*}(\delta) = s_{kjn^*}(\chi) = 0$). In this case, the $\mathcal{G}$ symbol is equivalent to the G symbol, and the Θ symbol also has the simple form:

$$\Theta^{kji,\alpha\beta} = O^{jk}_{i^*} A^i \delta_{\alpha\beta}, \tag{113}$$

From the pentagon equations of F:

$$\sum_{\epsilon} F^{mkn,\beta\chi}_{lpq,\delta\epsilon} F^{ijm,\alpha\epsilon}_{qps,\phi\gamma} = \sum_{t}\sum_{\eta,\varphi,\kappa} F^{ijm,\alpha\beta}_{knt,\eta\varphi} F^{itn,\varphi\chi}_{lps,\kappa\gamma} F^{jkt,\eta\kappa}_{lsq,\delta\phi} \tag{114}$$

we have:

$$\sum_{\epsilon} \frac{G^{mkn,\beta\chi}_{lpq,\delta\epsilon}}{O^{pm^*}_{q} O^{k^*l^*}_{q^*} A^{q^*}} \frac{G^{ijm,\alpha\epsilon}_{qps,\phi\gamma}}{O^{pi^*}_{s} O^{j^*q^*}_{s^*} A^{s^*}} \tag{115}$$

$$= \sum_{t}\sum_{\eta,\varphi,\kappa} \frac{G^{ijm,\alpha\beta}_{knt,\eta\varphi}}{O^{ni^*}_{t} O^{j^*k^*}_{t^*} A^{t^*}} \frac{G^{itn,\varphi\chi}_{lps,\kappa\gamma}}{O^{pi^*}_{s} O^{t^*l^*}_{s^*} A^{s^*}} \frac{G^{jkt,\eta\kappa}_{lsq,\delta\phi}}{O^{sj^*}_{q} O^{k^*l^*}_{q^*} A^{q^*}}.$$

Using the cyclic relation (112), we further have:

$$\sum_{\epsilon} \frac{G^{mkn,\beta\chi}_{lpq,\delta\epsilon} G^{ijm,\alpha\epsilon}_{qps,\phi\gamma}}{O^{pm^*}_{q} A^{q^*}} = \sum_{t}\sum_{\eta,\varphi,\kappa} \frac{G^{ijm,\alpha\beta}_{knt,\eta\varphi} G^{itn,\varphi\chi}_{lps,\kappa\gamma} G^{jkt,\eta\kappa}_{lsq,\delta\phi}}{O^{ni^*}_{t} O^{j^*k^*}_{t^*} A^{t^*} O^{t^*l^*}_{s^*} A^{s^*}}$$

$$\sum_{\epsilon} \frac{G^{mkn,\beta\chi}_{lpq,\delta\epsilon} G^{ijm,\alpha\epsilon}_{qps,\phi\gamma}}{O^{pm^*}_{q} A^{q^*}} = \sum_{t}\sum_{\eta,\varphi,\kappa} \frac{A^t G^{ijm,\alpha\beta}_{knt,\eta\varphi} G^{itn,\varphi\chi}_{lps,\kappa\gamma} G^{jkt,\eta\kappa}_{lsq,\delta\phi}}{O^{ni^*}_{t} A^{t^*} O^{j^*k^*}_{t^*} A^{t} O^{t^*l^*}_{s^*} A^{s}}. \tag{116}$$

In the last equation, we use the condition $A^i = A^{i^*}$.

In terms of Θ symbol, we can further simplify the above equation as:

$$\sum_{\epsilon\epsilon'} G^{mkn,\beta\chi}_{lpq,\delta\epsilon} G^{ijm,\alpha\epsilon'}_{qps,\phi\gamma} \left(\Theta^{m^*pq^*,\epsilon\epsilon'}\right)^{-1}$$
$$= \sum_{t}\sum_{\eta,\varphi,\kappa,\eta',\varphi',\kappa'} A^t G^{ijm,\alpha\beta}_{knt,\eta\varphi} G^{itn,\varphi'\chi}_{lps,\kappa\gamma} G^{jkt,\eta'\kappa'}_{lsq,\delta\phi} \left(\Theta^{i^*nt^*,\varphi\varphi'}\right)^{-1} \left(\Theta^{k^*j^*t,\eta\eta'}\right)^{-1} \left(\Theta^{l^*t^*s,\kappa\kappa'}\right)^{-1}. \tag{117}$$

The generalization to the fermionic $2 \to 3/3 \to 2$ move is straightforward and we only need to replace the G and Θ symbol with their corresponding fermionic version:

$$\sum_{\epsilon\epsilon'} \mathcal{G}^{mkn,\beta\chi}_{lpq,\delta\epsilon} \mathcal{G}^{ijm,\alpha\epsilon'}_{qps,\phi\gamma} \left(\mathbb{\Theta}^{m^*pq^*,\epsilon\epsilon'}\right)^{-1}$$
$$= \sum_{t}\sum_{\eta,\varphi,\kappa,\eta',\varphi',\kappa'} A^t \mathcal{G}^{ijm,\alpha\beta}_{knt,\eta\varphi} \mathcal{G}^{itn,\varphi'\chi}_{lps,\kappa\gamma} \mathcal{G}^{jkt,\eta'\kappa'}_{lsq,\delta\phi} \left(\mathbb{\Theta}^{i^*nt^*,\varphi\varphi'}\right)^{-1} \left(\mathbb{\Theta}^{k^*j^*t,\eta\eta'}\right)^{-1} \left(\mathbb{\Theta}^{l^*t^*s,\kappa\kappa'}\right)^{-1}. \tag{118}$$

Notice for the fermionic $\mathbb{\Theta}^{-1}$ symbol, we need to apply the right hand rule to determine the ordering of the Majorana numbers. Actually this is crucial to derive the correct fermionic pentagon equation for $\mathcal{F}$:

$$\sum_{\epsilon} \mathcal{F}^{mkn,\beta\chi}_{lpq,\delta\epsilon} \mathcal{F}^{ijm,\alpha\epsilon}_{qps,\phi\gamma} = \sum_{t}\sum_{\eta,\varphi,\kappa} \mathcal{F}^{ijm,\alpha\beta}_{knt,\eta\varphi} \mathcal{F}^{itn,\varphi\chi}_{lps,\kappa\gamma} \mathcal{F}^{jkt,\eta\kappa}_{lsq,\delta\phi}.$$

In terms of $\mathcal{F}$, the ordering is fixed because they do *not* commute with each other. However, in terms of $\mathcal{G}$ symbol in eqn. (118), we don't need to specify the ordering since all of them carry different Majorana numbers, but we do need to put all the $\mathcal{G}$ symbol in front of Θ symbol.

From the graphic representation of the $\mathcal{G}$ symbol, it is straightforward to represent the above generalized pentagon identity of the $\mathcal{G}$ symbol as a two tetrahedron to three tetrahedron move(and its inverse), see in Fig. 9.

D. The $2 \to 0/0 \to 2$ move for the $\mathcal{G}$ symbol

1. *The associativity condition of A^i*

To derive the $2 \to 0/0 \to 2$ move for the $\mathcal{G}$ symbol, we need to further impose another associativity condition for A^i:

$$\sum_{ij\alpha} \delta_{k^*ij} A^i A^j (-1)^{s_{k^*ij}(\alpha)} = DA^k, \tag{119}$$

where D is a constant. Notice the factor $(-1)^{s_{k^*ij}(\alpha)}$ is a new feature uniquely arises from our fermionic generalization. In the bosonic case, this equation can be simplified as:

$$\sum_{ij\alpha} \delta_{k^*ij} A^i A^j = DA^k, \tag{120}$$

The loop value A^i is proportional to the quantum dimension, which should be a positive real number for most unitary theories. However, A^i can be negative in

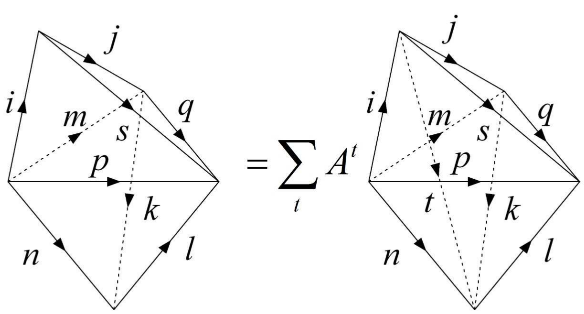


FIG. 9. Graphic representation of the pentagon identity for $\mathcal{G}$ symbol, Each tetrahedra represents a $\mathcal{G}$ symbol. When two tetrahedral share a face, we insert the Θ^{-1} symbol as a metric and trace over both the front face and opposite face indices for that face. In this figure we omit the face indices for simplification.

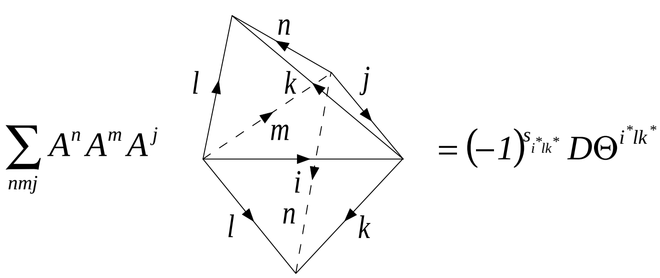


FIG. 10. Graphic representation of the $2 \to 0/0 \to 2$ move for $\mathcal{G}$ symbol. In this figure we also omit the face indices for simplification.

our fermionic generalization and we believe the factor $(-1)^{s_{k^*ij}(\alpha)}$ play an important role to maintain the unitary properties of the theory. Such a conjuncture of the associativity relation on A^i is consistent with other conditions but the underlying mathematical structure is still unclear, which will be studied in details in our future publications.

2. *The $2 \to 0/0 \to 2$ move*

To derive the $2 \to 0/0 \to 2$ move, let us again start from the simple bosonic case. In this case, under the orthogonality condition and tetrahedral symmetry condition, we can simplify the $\mathcal{F}, \mathcal{O}$ relation Eq. (74) equation as:

$$O_i^{jm}\delta_{il^*k}\delta_{\alpha\eta}\delta_{ip}\delta_{\beta\delta} = \sum_{n\chi\lambda} F^{mjp,\eta\beta}_{kln,\chi\lambda} F^{ij^*m,\alpha\lambda}_{nlk,\chi\delta} O^{jn^*}_{k^*} \tag{121}$$

In terms of G, we have:

$$\frac{O_i^{jm} A^{i^*}\delta_{il^*k}\delta_{ip}\delta_{\alpha\eta}\delta_{\beta\delta}}{A^{i^*}} = \sum_{n\chi\lambda} \frac{A^n G^{mjp,\eta\beta}_{kln,\chi\lambda} G^{ij^*m,\alpha\lambda}_{nlk,\chi\delta}}{O^{li^*}_{k} A^{k^*} O^{lm*}_{n} A^{n^*} O^{j^*k^*}_{n^*} A^n} \tag{122}$$

Use the definition of Θ symbol, we further have:

$$\frac{\delta_{ip}}{A^{i^*}} \Theta^{mji^*,\eta\alpha}\Theta^{i^*lk^*,\delta\beta} = \sum_{n\chi\lambda\chi'\lambda'} A^n G^{mjp,\eta\beta}_{kln,\chi\lambda} G^{ij^*m,\alpha\lambda'}_{nlk,\chi'\delta} \left(\Theta^{k^*j^*n,\chi\chi'}\right)^{-1} \left(\Theta^{m^*ln^*,\lambda\lambda'}\right)^{-1} \tag{123}$$

Combining with the associativity equation for A^i, we can derive the $2 \to 0/0 \to 2$ move for G symbol (see Fig. 10):

$$D\Theta^{i^*lk^*,\delta\beta} = \sum_{nmj\chi\chi'\lambda\lambda'\eta\alpha} A^n A^m A^j G^{mji,\eta\beta}_{kln,\chi\lambda} G^{ij^*m,\alpha\lambda'}_{nlk,\chi'\delta} \times \left(\Theta^{k^*j^*n,\chi\chi'}\right)^{-1} \left(\Theta^{m^*ln^*,\lambda\lambda'}\right)^{-1} \left(\Theta^{mji^*,\eta\alpha}\right)^{-1}. \tag{124}$$

Similar to the $2 \to 3/3 \to 2$ move, we can generalize the above expression to the fermionic case by replacing $\mathcal{G}$ and Θ with the corresponding fermionic version. However, due to the anti-commutative relation between those Majorana number, a simple replacing does not work. Let us start from Eq. (74) and treat the sign carefully. Again, under the orthogonality condition of $\mathcal{O}$, we have:

$$O_i^{jm}\delta_{il^*k}\delta_{\alpha\eta}\delta_{ip}\delta_{\beta\delta}\theta_{\underline{\alpha}}^{s_{mji^*}(\alpha)}\theta_{\underline{\eta}}^{s_{mji^*}(\eta)}\theta_{\underline{\beta}}^{s_{kl^*i}(\beta)}\theta_{\underline{\delta}}^{s_{kl^*i}(\delta)} = \sum_{n\chi\lambda} \mathcal{F}^{mjp,\eta\beta}_{kln,\chi\lambda}\mathcal{F}^{ij^*m,\alpha\lambda}_{nlk,\chi\delta}\mathcal{O}^{jn^*}_{k^*}. \tag{125}$$

In terms of $\mathcal{G}$, we have:

$$\frac{O_i^{jm} A^{i^*}\delta_{il^*k}\delta_{ip}\delta_{\alpha\eta}\delta_{\beta\delta}\theta_{\underline{\alpha}}^{s_{mji^*}(\alpha)}\theta_{\underline{\eta}}^{s_{mji^*}(\eta)}\theta_{\underline{\beta}}^{s_{kil^*}(\beta)}\theta_{\underline{\delta}}^{s_{kil^*}(\delta)}}{A^{i^*}} = \sum_{n\chi\lambda} \frac{A^n \mathcal{G}^{mjp,\eta\beta}_{kln,\chi\lambda}\mathcal{G}^{ij^*m,\alpha\lambda}_{nlk,\chi\delta}}{O^{li^*}_{k} A^{k^*} O^{lm*}_{n} A^{n^*} O^{j^*k^*}_{n^*} A^n}. \tag{126}$$

Recall the definition of Θ, we can derive:

$$\frac{\delta_{ip}}{A^{i^*}}(-1)^{s_{mji^*}(\alpha)}(-1)^{s_{i^*lk^*}(\beta)}\Theta^{mji^*,\eta\alpha}\Theta^{i^*lk^*,\delta\beta} = \sum_{n\chi\lambda\chi'\lambda'} A^n G^{mjp,\eta\beta}_{kln,\chi\lambda} G^{ij^*m,\alpha\lambda'}_{nlk,\chi'\delta} \left(\Theta^{k^*j^*n,\chi\chi'}\right)^{-1} \left(\Theta^{m^*ln^*,\lambda\lambda'}\right)^{-1} \tag{127}$$

Notice the sign factors $(-1)^{s_{mji^*}(\alpha)}$ and $(-1)^{s_{i^*lk^*}(\beta)}$ arise from the ordering of Majorana number.

Combining with the fermionic associativity relation of A^i Eq. (119), we finally have:

$$(-1)^{s_{i^*lk^*}(\beta)} D\Theta^{i^*lk^*,\delta\beta} = \sum_{nmj\chi\chi'\lambda\lambda'\eta\alpha} A^nA^mA^j \mathcal{G}^{mji,\eta\beta}_{kln,\chi\lambda}\mathcal{G}^{ij^*m,\alpha\lambda'}_{nlk,\chi'\delta}\times \left(\Theta^{k^*j^*n,\chi\chi'}\right)^{-1}\left(\Theta^{m^*ln^*,\lambda\lambda'}\right)^{-1}\left(\Theta^{mji^*,\eta\alpha}\right)^{-1}. \tag{128}$$

Same as in the $2\to 3/3\to 2$ move, the ordering of $\mathcal{G}$ symbol is not important as long as they appear in front of Θ^{-1} symbols. However, in terms of $\mathcal{F},\mathcal{O}$, the ordering is important and it is automatically taken care of by the ordering of Majorana number in Θ^{-1} symbol, which is determined by the left hand rule.

E. Topologically invariant partition function

Based on the $3\to 2/2\to 3$ move and $2\to 0/0\to 2$ move, we can prove the following partition function is independent of the triangulation of closed triangulated 3-manifolds.[39–41]

$$Z_{top} = \frac{1}{D^{N_v}}\sum_{\alpha\beta\gamma\delta\cdots}\sum_{ijklmn\cdots} \prod_{\text{link}} A^i \prod_{\text{tetrahedron}} \mathcal{G}^{ijm,\alpha\beta}_{kln,\gamma\delta}\prod_{\text{face}}\left(\Theta^{ijm^*,\alpha\alpha'}\right)^{-1}, \tag{129}$$

where N_v is the number of vertex.

From the $2\to 3/3\to 2$ move, it is obvious the above partition function is a invariance for those triangulations with the same number of vertices. To prove the invariance still holds for those triangulations with different number of vertices, we only need to show the correctness of $4\to 1/1\to 4$ move based on the $2\to 3/3\to 2$ move and $2\to 0/0\to 2$ move. The proof is very similar as the bosonic case, however, due to the fermionic nature, we need to take care of the sign. See in Fig 11, we want to prove:

$$\begin{aligned}&\sum_{\eta\eta'\chi\chi'\lambda\lambda'\phi\phi'\kappa\kappa'\varphi\varphi'}\sum_{nmjt} A^nA^mA^jA^t \\ &\times \mathcal{G}^{mji,\eta\beta}_{kln,\chi\lambda}\mathcal{G}^{mnl,\lambda'\gamma}_{qst,\kappa\phi}\mathcal{G}^{ij^*m,\eta'\phi'}_{tsp,\varphi'\sigma}\mathcal{G}^{jkn,\chi'\kappa'}_{qtp,\rho\varphi}\\ &\times\left(\Theta^{tq^*n^*,\kappa\kappa'}\right)^{-1}\left(\Theta^{tpj^*,\varphi\varphi'}\right)^{-1}\left(\Theta^{m^*st^*,\phi\phi'}\right)^{-1}\\ &\times\left(\Theta^{mji^*,\eta\eta'}\right)^{-1}\left(\Theta^{k^*j^*n,\chi\chi'}\right)^{-1}\left(\Theta^{m^*ln^*,\lambda\lambda'}\right)^{-1}\\ &= D\mathcal{G}^{ikl,\beta\gamma}_{qsp,\rho\sigma}\end{aligned} \tag{130}$$

By applying the $3\to 2$ move, the left side of the above equation becomes

$$\sum_{\eta\eta'\chi\chi'\lambda\lambda'\delta\delta'}\sum_{nmj} A^nA^mA^j\mathcal{G}^{mji,\eta\beta}_{kln,\chi\lambda}\mathcal{G}^{ij^*m,\eta'\lambda'}_{nlk,\chi'\delta}\mathcal{G}^{ikl,\beta\gamma}_{qsp,\rho\sigma} \times\left(\Theta^{mji^*,\eta\eta'}\right)^{-1}\left(\Theta^{k^*j^*n,\chi\chi'}\right)^{-1}\left(\Theta^{m^*ln^*,\lambda\lambda'}\right)^{-1}\left(\Theta^{i^*lk^*,\delta\delta'}\right)^{-1} \tag{131}$$

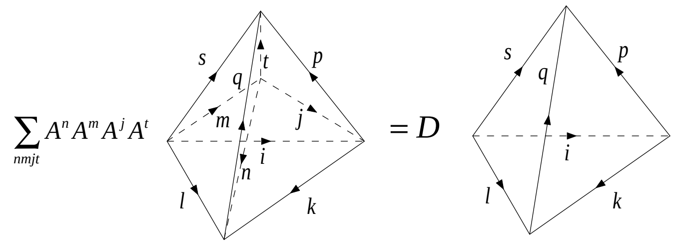


FIG. 11. Graphic representation of the $4\to 1/1\to 4$ move for $\mathcal{G}$ symbol. In this figure we also omit the face indices for simplification.

Finally, we can apply the $2\to 0$ move and further simplify the above expression as:

$$\begin{aligned}&\sum_{\delta\delta'} D(-1)^{s_{i^*lk^*}(\beta)}\mathcal{G}^{ikl,\beta\gamma}_{qsp,\rho\sigma}\Theta^{i^*lk^*,\delta\beta}\left(\Theta^{i^*lk^*,\delta\delta'}\right)^{-1}\\ &=\sum_{\delta\delta'} D\mathcal{G}^{ikl,\beta\gamma}_{qsp,\rho\sigma}(-1)^{s_{i^*lk^*}(\beta)}\theta^{s_{kl^*i}(\beta)}_{\underline{\beta}}\theta^{s_{kl^*i}(\delta')}_{\underline{\delta'}}\delta_{\delta\beta}\delta_{\delta\delta'}\\ &=\sum_{\delta'} D\mathcal{G}^{ikl,\beta\gamma}_{qsp,\rho\sigma}\theta^{s_{kl^*i}(\delta')}_{\underline{\delta'}}\theta^{s_{kl^*i}(\beta)}_{\underline{\beta}}\delta_{\delta'\beta}\\ &= D\mathcal{G}^{ikl,\beta\gamma}_{qsp,\rho\sigma}\end{aligned} \tag{132}$$

The inverse of the above $4\to 1$ move is also true, thus we proof the invariance of the partition function Eq. (129) for those triangulations with different number of vertices.

We believe such a Majorana valued partition function can describe almost all the quantum double of spin Chen-Simons theory.[42]

F. Simplified fixed-point conditions

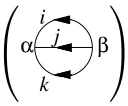
We have seen that under certain extra conditions, such as the tetrahedral conditions (100) and the orthogonality condition (102), the F-tensors and O-tensors do not depend on the ordering of vertices: $F^{ijm,\alpha\beta,a}_{kln,\chi\delta,b} = F^{ijm,\alpha\beta}_{kln,\chi\delta}$, $O^{jk,\alpha\beta}_{i,a} = O^{jk,\alpha\beta}_{i}$. In this case eqn. (92), eqn. (93), and eqn. (94) can be simplified. We find that

- $\sum_{n\chi\delta} F^{ijm',\alpha'\beta'}_{kln,\chi\delta}(F^{ijm,\alpha\beta}_{kln,\chi\delta})^* = \delta_{m\alpha\beta,m'\alpha'\beta'}$,
- $F^{ijm,\alpha\beta}_{kln,\chi\delta} = 0$ when $N_{jim^*}<1$ or $N_{kml^*}<1$ or $N_{kjn^*}<1$ or $N_{nil^*}<1$, or $s_{jim^*}(\alpha)+s_{kml^*}(\beta)+s_{kjn^*}(\chi)+s_{nil^*}(\delta) =$ odd,
- $$\sum_t\sum_{\eta=1}^{N_{kjt^*}}\sum_{\varphi=1}^{N_{tin^*}}\sum_{\kappa=1}^{N_{lts^*}} F^{ijm,\alpha\beta}_{knt,\eta\varphi}F^{itn,\varphi\chi}_{lps,\kappa\gamma}F^{jkt,\eta\kappa}_{lsq,\delta\phi} = (-)^{s_{jim^*}(\alpha)s_{lkq^*}(\delta)}\sum_{\epsilon=1}^{N_{qmp^*}} F^{mkn,\beta\chi}_{lpq,\delta\epsilon}F^{ijm,\alpha\epsilon}_{qps,\phi\gamma}. \tag{133}$$

- $\sum_{k,j} \sum_{\alpha=1}^{N_{kii^*}} \sum_{\beta=1}^{N_{j^*jk^*}} O_i^{jk,\alpha\beta} (O_i^{jk,\alpha\beta})^* = 1,$
- $O_i^{jk,\alpha\beta} = 0, \text{ if } N_{ik^*j^*} < 1, \text{ or } N_{i^*jk} < 1,$ or $s_{ik^*j^*}(\alpha) + s_{i^*jk}(\beta) = \text{ odd},$
- $\mathrm{e}^{\mathrm{i}\theta_{O1}} O_i^{jm,\alpha\eta} \delta_{ip}\delta_{\beta\delta} = \sum_{n\chi\lambda\gamma} (-)^{s_{kjn^*}(\chi)[s_{kjn^*}(\chi)+s_{jmp^*}(\eta)]} \times F^{mjp,\eta\beta}_{kln,\chi\lambda} F^{ij^*m,\alpha\lambda}_{nlk,\gamma\delta} O^{jn^*,\gamma\chi}_{k^*}$

for all k, i, l satisfying $N_{kil^*} > 0$. (134)

- $\sum_i A^i (A^i)^* = 1,$ • $A^i = A^{i^*} \neq 0,$
- $O_i^{j^*k^*,\alpha\beta} A^{i^*} = O_j^{k^*i^*,\alpha\beta} A^j,$
- $\det(M_{kji,\,\underline{\alpha\beta}}) \equiv \det \left[\psi_{\text{fix}} \left(\right) \right] \neq 0.$ (135)



VII. CATEGORICAL FRAMEWORK

To provide a conceptual understanding of our generalization of string-net model, we discuss briefly the categorical picture which underlies earlier algebraic manipulations. Such a mathematical framework will provide more examples for our fermionic string-net Hamiltonians in Appendix B.

A string-net or Levin-Wen Hamiltonian can be easily constructed using $6j$-symbols from a unitary fusion category $\mathcal{C}$. The elementary excitations of the model form a unitary modular tensor category (UMTC) $\mathcal{E}$, which turns out to be the quantum double $Z(\mathcal{C})$ of the input category $\mathcal{C}$. A priori, the output modular category $\mathcal{E}$ is not necessarily related to the input category $\mathcal{C}$. Therefore, it is conceivable that similar Hamiltonians can be constructed from some other algebraic data where the elementary excitations still form a UMTC, which is not necessarily a quantum double. This is explored in Ref. 6. In the preceding sections, we generalize the string-net model by including fermionic degrees of freedom.

The mathematical framework for such a generalization is the theory of enriched categories.[24] An enriched category is actually not a category, just like a quantum group is not a group. We will consider only special enriched categories, which we call *projective super fusion categories.* The ordinary unitary fusion categories are enriched categories over the category of Hilbert spaces, while projective super fusion categories are enriched categories over the category of super Hilbert spaces up to *projective even* unitary transformations.

To the physically inclined readers, the use of category theory in condensed matter physics seems to be unjustifiably abstract. We would argue that the abstractness of category theory is actually its virtue. Topological properties of quantum systems are independent of the microscopic details and non-local. A framework to encode such properties is necessarily blind to microscopic specifics. Therefore, philosophically category theory could be extremely relevant, as we believe.

A. Projective super tensor category

We use super vector spaces to accommodate fermionic states, and generalize the composition of linear transformations to one only up to overall phases—a possibility allowed by quantum mechanics. The projective tensor category of vector spaces is the category of vector spaces and linear transformations composed up to overall phases, and the category of super vector spaces is the tensor category of $\mathbb{Z}_2$-graded vector spaces and all *even* linear transformations.

In the categorical language, a fusion category is a rigid finite linear category with a simple unit. Equivalently, it can be defined using $6j$-symbols: an equivalence class of solutions of pentagons satisfying certain normalizations[26]. Fermionic $6j$-symbols $\mathcal{F}^{ijm,\alpha\beta,a}_{kln,\gamma\lambda,b}$ in eqn. (30) with certain normalizations define a projective super fusion category if they satisfy fermionic pentagon equations eqn. (52). However, the setup used in this paper may only generate a subclass of projective super fusion category.

B. Super tensor category from super quantum groups

The trivial example of a super tensor category is the category of $\mathbb{Z}_2$-graded vector spaces and all linear transformations. More interesting examples of super tensor categories can be constructed from the representation theory of super quantum groups.

Super quantum groups are deformations of Lie superalgebras.[43,44] Though a mathematical theory analogous to quantum group exists, the details have not been worked out enough for our application here. In literature, the categorical formulation focuses on the invariant spaces of even entwiners, while for our purpose, we need to consider all entwiners. In particular, we are not aware of work on Majorana valued Clebsch-Gordon coefficients, therefore, we will leave the details to future publications.

VIII. SIMPLE SOLUTIONS OF THE FIXED-POINT CONDITIONS

In this section, let us discuss some simple solutions of the fixed-point conditions (91, 92, 93) for the fixed-point gfLU transformations $(N_{ijk}, N^f_{ijk}, s_i, F^{ijm,\alpha\beta,a}_{kln,\gamma\lambda,b}, O^{kj,\alpha\beta}_{i,a})$.

A. Unimportant phase factors in the solutions

Formally, the solutions of the fixed-point conditions are not isolated. They are parameterized by several continuous phase factors. In this section, we will discuss the origin of those phase factors. We will see that those different phase factors do not correspond to different states of matter (*ie* different equivalence classes of gfLU transformations). So after removing those unimportant phase factors, the solutions of the fixed-point conditions are isolated (at least for the simple examples studied here).

We notice that, apart from two normalization conditions, all of the fixed-point conditions are linear in $O^{jk,\alpha\beta}_{i,a}$. Thus if $(F^{ijm,\alpha\beta,a}_{kln,\gamma\lambda,b}, O^{jk,\alpha\beta}_{i,a})$ is a solution, then $(F^{ijm,\alpha\beta,a}_{kln,\gamma\lambda,b}, \mathrm{e}^{\mathrm{i}\varphi_a} O^{jk,\alpha\beta}_{i,a})$ is also a solution. However, the two phase factors, $\mathrm{e}^{\mathrm{i}\varphi_a}$, $a=\pm$ do not lead to different fixed-point wave functions, since they only affect the total phase of the wave function and are unphysical. Thus the total phases of $O^{jk,\alpha\beta}_{i,+}$ and $O^{jk,\alpha\beta}_{i,-}$ can be adjusted. We can use this degree of freedom to set, say, $O^{00,11}_{0,\pm} \geq 0$.

Similarly the total phase of $F^{ijm,\alpha\beta,\pm}_{kln,\gamma\lambda,\pm}$ is also unphysical and can be adjusted. We have used this degree of freedom to obtain eqn. (42) and eqn. (54) which do not contain the over all phase factor.

The above two phase factors are unphysical. However, the fixed-point solutions may also contain phase factors that do correspond to different fixed-point wave functions. For example, the local unitary transformation $\mathrm{e}^{\mathrm{i}\theta_{l_0}\hat{M}_{l_0}}$ does not affect the fusion rule N_{ijk}, where $\hat{M}_{l_0}$ is the number of edges with $|l_0\rangle$-state and $|l_0^*\rangle$-state. Such a local unitary transformation changes $(F^{ijm,\alpha\beta,a}_{kln,\gamma\lambda,b}, O^{kj,\alpha\beta}_{i,a})$ and generates a continuous family of the fixed-point wave functions parameterized by θ_{l_0}. Those wave functions are related by local unitary transformations that continuously connect to identity. Thus, those fixed-point wave functions all belong to the same phase.

Similarly, we can consider the following local unitary transformation $|\alpha\rangle \to \sum_\beta U^{(i_0 j_0 k_0)}_{\alpha\beta}|\beta\rangle$ that acts on each vertex with states $|i_0\rangle, |j_0\rangle, |k_0\rangle$ on the three edges connecting to the vertex. Such a local unitary transformation also does not affect the fusion rule N_{ijk}. The new local unitary transformation changes $(F^{ijm,\alpha\beta,a}_{kln,\gamma\lambda,b}, O^{kj,\alpha\beta}_{i,a})$ and generates a continuous family of the fixed-point wave functions parameterized by the unitary matrix $U^{(i_0 j_0 k_0)}_{\alpha\beta}$. Again, those fixed-point wave functions all belong to the same phase.

In the following, we will study some simple solutions of the fixed-point conditions. We find that, for those examples, the solutions have no additional continuous parameters apart from the phase factors discussed above. This suggests that the solutions of the fixed-point conditions correspond to isolated zero-temperature phases.

B. A $\mathbb{Z}_3$ bosonic fixed-point

Let us consider a system where there are three states $|0\rangle$, $|1\rangle$, and $|2\rangle$ on each edge of the graph. We choose

$$0^* = 0, \quad 1^* = 2, \quad 2^* = 1. \tag{136}$$

The simplest bosonic fusion rule that satisfies (91) is

$$\begin{aligned} &N_{000} = N_{111} = N_{222} = N_{120} = N_{201} = N_{012} \\ &= N_{210} = N_{021} = N_{102} = 1, \\ &\text{other } N_{ijk} = 0, \\ &N^f_{ijk} = 0. \end{aligned} \tag{137}$$

Since $N_{ijk} \leq 1$, there is no state on the vertices. So the indices $\alpha, \beta, ...$ labeling the states on a vertex can be suppressed. The above fusion rule corresponds to the fusion rule for the string-net state that describes the $\mathbb{Z}_3$ gauge theory.[19] So we will call the corresponding graphic state $\mathbb{Z}_3$ bosonic state (or simply $\mathbb{Z}_3$ state), since $N^f_{ijk} = 0$.

After fixing the unimportant phase factors, we find that the fixed-point conditions have only one isolated solution:

$$F^{000,+}_{000,-} = F^{000,+}_{111,-} = F^{000,+}_{222,-} = 1$$
$$F^{011,+}_{011,-} = F^{011,+}_{122,-} = F^{011,+}_{200,-} = 1$$
$$F^{022,+}_{022,-} = F^{022,+}_{100,-} = F^{022,+}_{211,-} = 1$$
$$F^{101,+}_{010,-} = F^{101,+}_{121,-} = F^{101,+}_{202,-} = 1$$
$$F^{112,+}_{021,-} = F^{112,+}_{102,-} = F^{112,+}_{210,-} = 1$$
$$F^{120,+}_{002,-} = F^{120,+}_{110,-} = F^{120,+}_{221,-} = 1$$
$$F^{202,+}_{020,-} = F^{202,+}_{101,-} = F^{202,+}_{212,-} = 1$$
$$F^{210,+}_{001,-} = F^{210,+}_{112,-} = F^{210,+}_{220,-} = 1$$
$$F^{221,+}_{012,-} = F^{221,+}_{120,-} = F^{221,+}_{201,-} = 1$$

$$\begin{aligned} &O^{00}_{0,+} = O^{01}_{1,+} = O^{10}_{1,+} = O^{02}_{2,+} = O^{11}_{2,+} = O^{20}_{2,+} = \frac{1}{\sqrt{3}} \\ &O^{12}_{0,+} = O^{21}_{0,+} = O^{22}_{1,+} = \frac{1}{\sqrt{3}}. \end{aligned} \tag{138}$$

We note that $F^{ijm,\alpha\beta,a}_{kln,\chi\delta,b}$ in eqn. (25) relates wave functions on two graphs. In the above, we have drawn the two related graphs next to the F tensor, where the first graph following the F corresponds to the graph on the left-hand side of eqn. (25) and the second graph corresponds to the graph on the right-hand side of eqn. (25). The doted line

corresponds to the $|0\rangle$-state on the edge and the solid line corresponds to the $|1\rangle$-state or the $|2\rangle$-state on the edge, depending on the direction of the edge.

$F^{ijm,a}_{kln,b}$ can be determined from $F^{ijm,+}_{kln,-}$ (see eqn. (44)). It turns out that $F^{ijm,a}_{kln,b} = F^{ijm,+}_{kln,-}$ for the $\mathbb{Z}_3$ state. Similarly, $O^{jk,\alpha\beta}_{i,-}$ can be determined from $O^{jk,\alpha\beta}_{i,+}$ (see eqn. (67)) and we find $O^{jk,\alpha\beta}_{i,-} = O^{jk,\alpha\beta}_{i,+}$. Also the phase factors are all equal to 1 for the $\mathbb{Z}_3$ state:

$$e^{i\theta^{ab}_F} = e^{i\theta_{O1}} = e^{i\theta_{O2}} = 1. \tag{139}$$

We have introduced A^i to describe the fixed-point wave function. The $\mathbb{Z}_3$ fixed-point wave function is described by

$$A^0 = A^1 = A^2 = \frac{1}{\sqrt{3}}. \tag{140}$$

The above solution $(F^{ijm,\alpha\beta,a}_{kln,\gamma\lambda,b}, O^{jk,\alpha\beta}_{i,a})$ gives rise to a fixed-point state which is an equal weight superposition of all the graphic states that satisfy the fusion rule $N_{ijk} = 1$ on every vertex, where $(|i\rangle, |j\rangle, |k\rangle)$ are the three states on the edges that connect to the vertex. If we view the state $|1\rangle$ and $|2\rangle$ on a edge as a segment of string on the edge pointing in opposite directions and state $|0\rangle$ as no string state, then the graphic states that satisfy the fusion rule $N_{ijk} = 1$ can be viewed as states of orientable loops of strings. The fixed-point state is an equal weight superposition of all the loops of strings.

If we put fixed-point state on a torus (*ie* a planar graph with periodic boundary condition in both x- and y-directions), there are will be 9 states. This can be seen by viewing the orientable strings as flux lines carrying a flux quantum. The fusion rule N_{ijk} implies that the flux is conserved mod 3. So the total flux quanta flowing across a non-contractible circle of the torus can be 0,1,2. There are two non-contractible circles on a torus which leads to 9 states (see Fig. 12).

The $\mathbb{Z}_3$ state has a topological order described by the $\mathbb{Z}_3$ gauge theory at low energies,[19,45] or equivalently by a $U(1) \times U(1)$ Chern-Simons (CS) theory:[46]

$$\mathcal{L} = \frac{1}{4\pi} K^{IJ} a_{I\mu} \partial_\nu a_{J\nu} \epsilon^{\mu\nu\lambda} + \ldots \tag{141}$$

with $K = \begin{pmatrix} 0 & 3 \\ 3 & 0 \end{pmatrix}$.

C. A $\mathbb{Z}_3$ fermionic fixed-point

Now let us discuss a fermionic example. We again consider a system with three states $|0\rangle$, $|1\rangle$, and $|2\rangle$ on each edge and we also choose

$$0^* = 0, \quad 1^* = 2, \quad 2^* = 1, \tag{142}$$

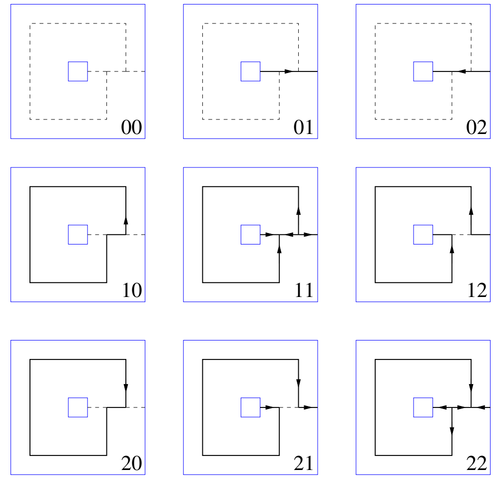


FIG. 12. (Color online) The 9 fixed-point states on a "simple torus". (The inner square and the outer square are identified to form a torus.) Any torus formed by more complex graph can always be reduced to the "simple torus" by the F-moves and the O-moves. The dotted lines represent the $|0\rangle$-state on the edges, and the oriented solid lines represent the $|1\rangle$-state or $|2\rangle$-state on the edges. The pair of integers mn at the lower right corner of each graph indicates the total flux quanta (mod 3) flowing across the two non-contractible circles of the torus.

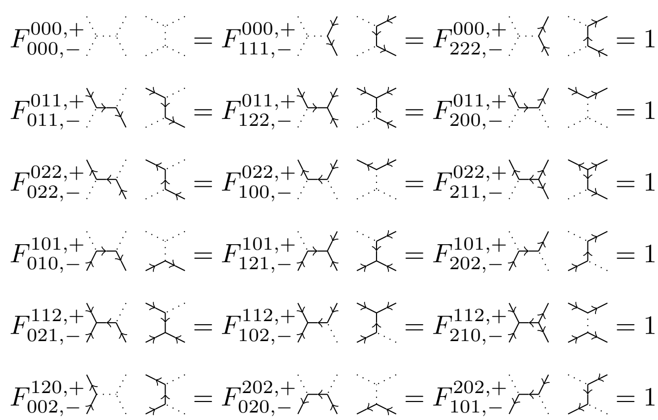
The simplest fermionic fusion rule that satisfies (91) is

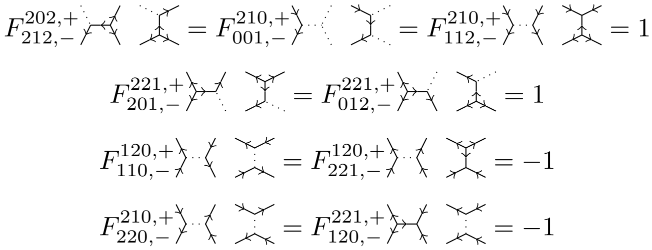
$$\begin{aligned} &N_{000} = N_{111} = N_{222} = N_{120} = N_{201} = N_{012} \\ &= N_{210} = N_{021} = N_{102} = 1, \\ &\text{other } N_{ijk} = 0. \end{aligned} \tag{143}$$

$$\begin{aligned} &N^f_{111} = N^f_{222} = 1, \\ &\text{other } N^f_{ijk} = 0. \end{aligned} \tag{144}$$

The above N_{ijk} is the same fusion rule for the $\mathbb{Z}_3$ string-net state discussed above. However, $N^f_{ijk} \neq 0$. So we will call the corresponding graphic state $\mathbb{Z}_3$ fermionic state.

After fixing the unimportant phase factors, we find that the fixed-point conditions have only one isolated solution:





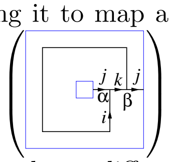
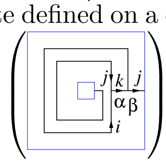
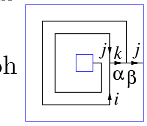
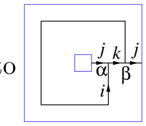
$$O^{00}_{0,+} = O^{01}_{1,+} = O^{10}_{1,+} = O^{02}_{2,+} = O^{11}_{2,+} = O^{20}_{2,+} = \frac{1}{\sqrt{3}}$$
$$O^{12}_{0,+} = O^{21}_{0,+} = O^{22}_{1,+} = -\frac{1}{\sqrt{3}} \tag{145}$$

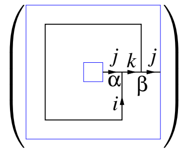
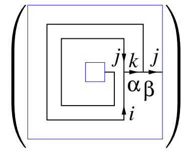
$$A^0 = \frac{1}{\sqrt{3}}, \qquad A^1 = A^2 = -\frac{1}{\sqrt{3}}. \tag{146}$$

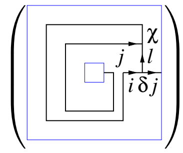
$F^{ijm,a}_{kln,b}$ can be determined from $F^{ijm,+}_{kln,-}$ (see eqn. (44)). It turns out that $F^{ijm,a}_{kln,b} = F^{ijm,+}_{kln,-}$ for the fermionic $\mathbb{Z}_3$ state. Similarly, $O^{jk,\alpha\beta}_{i,-}$ can be determined from $O^{jk,\alpha\beta}_{i,+}$ (see eqn. (67)) and we find $O^{jk,\alpha\beta}_{i,-} = O^{jk,\alpha\beta}_{i,+}$. Also the phase factors are all equal to 1 for the fermionic $\mathbb{Z}_3$ state: $\mathrm{e}^{\mathrm{i}\,\theta^{ab}_F} = \mathrm{e}^{\mathrm{i}\,\theta_{O1}} = \mathrm{e}^{\mathrm{i}\,\theta_{O2}} = 1$. Actually the above fermionic $\mathbb{Z}_3$ state also satisfies the tetrahedral conditions (100), the orthogonality condition (102) as well as the associativity condition (119). Thus such a solution is also an example of our fermionic generalization for Turaev-Viro invariance on closed triangulated 3-manifolds.

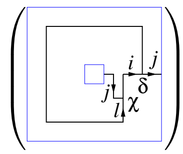
The above fermionic solution $(F^{ijm,\alpha\beta,a}_{kln,\gamma\lambda,b}, O^{jk,\alpha\beta}_{i,a})$ give rise to a fixed-point state which is a superposition of all the graphic states that satisfy fusion rule $N_{ijk} = 1$ on every vertex. However, the amplitude of each allowed graphic state can be $+1$ or -1. Using a similar argument, we can show that there are 9 fixed-point states on a torus, indicating that there are 9 types of quasiparticles for the fermionic $\mathbb{Z}_3$ state (see Fig. 12).

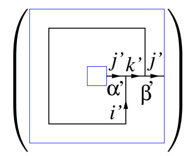
The fermionic $\mathbb{Z}_3$ state may be described by a $U(1)\times U(1)$ CS theory (141) with $K = \begin{pmatrix} 0 & 3 \\ 3 & m \end{pmatrix}$, and $m = 0, 1, 2, 3, 4, 5$. When m =even, the $U(1)\times U(1)$ CS theory describe a bosonic theory (such as the $m = 0$ case discussed before). In the following, we like to argue that the fermionic $\mathbb{Z}_3$ state corresponds to $m = 3$.

Let us consider a Dehn twist of the torus. Here we define the Dehn twist formaly by requiring it to map a fixed-point state $\psi^{\underline{\alpha}\,\underline{\beta}}_{\text{fix}}(i,j,k,\alpha,\beta) = \psi^{\underline{\alpha}\,\underline{\beta}}_{\text{fix}}$  on a torus to another fixed-point state defined on a different graph $\tilde\psi^{\underline{\alpha}\,\underline{\beta}}_{\text{fix}}(i,j,k,\alpha,\beta) = \tilde\psi^{\underline{\alpha}\,\underline{\beta}}_{\text{fix}}$  with $\psi^{\underline{\alpha}\,\underline{\beta}}_{\text{fix}}(i,j,k,\alpha,\beta) = \tilde\psi^{\underline{\alpha}\,\underline{\beta}}_{\text{fix}}(i,j,k,\alpha,\beta)$. Using a F-move we can deform the graph  to , which leads to an unitary transformation T between the fixed-point states on torus. The unitary matrix T can be calculated as the following:

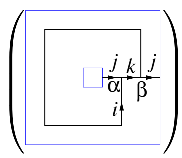
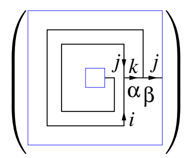
$\psi^{\underline{\alpha}\,\underline{\beta}}_{\text{fix}}$  $= \psi^{\underline{\alpha}\,\underline{\beta}}_{\text{fix}}$ 

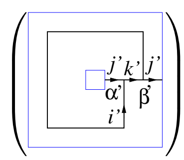
$= \sum_{l,\chi\delta} F^{ijk,\alpha\beta,\underline{\alpha\beta}}_{i^*jl^*,\chi\delta,\underline{\chi\delta}}\psi^{\underline{\chi}\,\underline{\delta}}_{\text{fix}}$  (147)

$= \sum_{l,\chi\delta} F^{ijk,\alpha\beta,\underline{\alpha\beta}}_{i^*jl^*,\chi\delta,\underline{\chi\delta}}\psi^{\underline{\chi}\,\underline{\delta}}_{\text{fix}}$ 

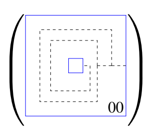
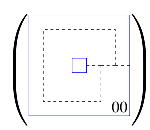
$= \sum_{i'j'k',\alpha'\beta'} \delta_{ik'}\delta_{jj'}F^{ijk,\alpha\beta,\underline{\alpha\beta}}_{i^*ji'^*,\alpha'\beta',\underline{\alpha'\beta'}}\psi^{\underline{\alpha'}\,\underline{\beta'}}_{\text{fix}}$  .

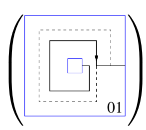
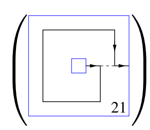
If we choose the vertex label as $\underline{\alpha} = \underline{\alpha'} = 1$ and $\underline{\beta} = \underline{\beta'} = 2$, we obtain

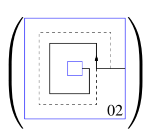
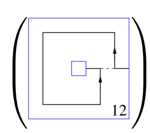
$\psi^{\underline{\alpha}\,\underline{\beta}}_{\text{fix}}$  $= \psi^{\underline{\alpha}\,\underline{\beta}}_{\text{fix}}$  (148)

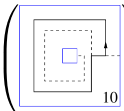
$= \sum_{i'j'k',\alpha'\beta'} \delta_{ik'}\delta_{jj'}F^{ijk,\alpha\beta,+}_{i^*ji'^*,\alpha'\beta',+}\psi^{\underline{\alpha'}\,\underline{\beta'}}_{\text{fix}}$  .

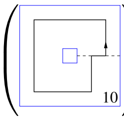
This allows us to obtain:

ψ_{fix}  $= F^{000,+}_{000,+}\psi_{\text{fix}}$ 

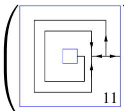
ψ_{fix}  $= F^{011,+}_{011,+}\psi_{\text{fix}}$ 

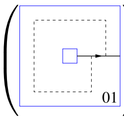
ψ_{fix}  $= F^{022,+}_{022,+}\psi_{\text{fix}}$ 

ψ_{fix} (

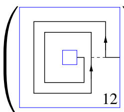
) $= F^{101,+}_{202,+}\psi_{\text{fix}}$ (

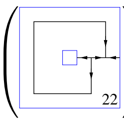
)

ψ_{fix} (

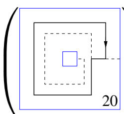
) $= F^{112,+}_{210,+}\psi_{\text{fix}}$ (

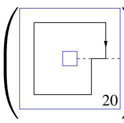
)

ψ_{fix} (

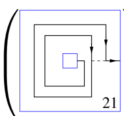
) $= (F^{120,+}_{221,+})\psi_{\text{fix}}$ (

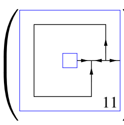
)

ψ_{fix} (

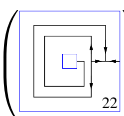
) $= F^{202,+}_{101,+}\psi_{\text{fix}}$ (

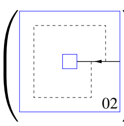
)

ψ_{fix} (

) $= F^{210,+}_{112,+}\psi_{\text{fix}}$ (

)

ψ_{fix} (

) $= (F^{221,+}_{120,+})\psi_{\text{fix}}$ (

)

where F's in ( ) are equal to -1 and other F's are equal to $+1$. We find that the 9 by 9 T-matrix for our fermionic $\mathbb{Z}_3$ state is

$$T = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \end{pmatrix} \tag{149}$$

For the bosonic states, the eigenvalues of T correspond to the statistical angles $\mathrm{e}^{\mathrm{i}\theta_i}$ of the corresponding quasiparticle.[8,23] For the fermionic states, we have a weaker result:[8] the eigenvalues of T^2 correspond to the square of the statistical angles, $\mathrm{e}^{2\mathrm{i}\theta_i}$.

The eigenvalues of the matrix T are

$$(1,1,1,1,1,\mathrm{e}^{\mathrm{i}2\pi/3},\mathrm{e}^{\mathrm{i}2\pi/3},\mathrm{e}^{\mathrm{i}4\pi/3},\mathrm{e}^{\mathrm{i}4\pi/3})$$

and the eigenvalues of the matrix T^2 are

$$(1,1,1,1,1,\mathrm{e}^{\mathrm{i}2\pi/3},\mathrm{e}^{\mathrm{i}2\pi/3},\mathrm{e}^{\mathrm{i}4\pi/3},\mathrm{e}^{\mathrm{i}4\pi/3}).$$

The $K = \begin{pmatrix} 0 & 3 \\ 3 & 3 \end{pmatrix}$ $U(1)\times U(1)$ CS theory has 9 types of quasiparticles with statistical angles

$$(\mathrm{e}^{\mathrm{i}\theta_i}) = (1,1,1,1,-1,\mathrm{e}^{\mathrm{i}\pi/3},\mathrm{e}^{\mathrm{i}4\pi/3},\mathrm{e}^{\mathrm{i}2\pi/3},\mathrm{e}^{\mathrm{i}5\pi/3}).$$

We note that $K = \begin{pmatrix} 0 & 3 \\ 3 & 3 \end{pmatrix}$ CS theory is equivalent to the $K = \begin{pmatrix} 3 & 0 \\ 0 & -3 \end{pmatrix}$ CS theory. However, for the $K = \begin{pmatrix} 3 & 0 \\ 0 & -3 \end{pmatrix}$ CS theory

$$(\mathrm{e}^{\mathrm{i}\theta_i}) = (1,1,1,-1,-1,\mathrm{e}^{\mathrm{i}\pi/3},\mathrm{e}^{\mathrm{i}4\pi/3},\mathrm{e}^{\mathrm{i}2\pi/3},\mathrm{e}^{\mathrm{i}5\pi/3}).$$

In fact quasiparticle statistical angles $\mathrm{e}^{\mathrm{i}\theta_i}$ always have $\pm$ ambiguity in fermionic CS theory, since the fermionic electron is reguared as "identity". However, $\mathrm{e}^{2\mathrm{i}\theta_i}$ is well defined. We see that

$$(\mathrm{e}^{2\mathrm{i}\theta_i}) = (1,1,1,1,1,\mathrm{e}^{\mathrm{i}2\pi/3},\mathrm{e}^{\mathrm{i}2\pi/3},\mathrm{e}^{\mathrm{i}4\pi/3},\mathrm{e}^{\mathrm{i}4\pi/3})$$

match the eigenvalues of T^2 exactly.

For the $K = \begin{pmatrix} 0 & 3 \\ 3 & 0 \end{pmatrix}$ $U(1)\times U(1)$ CS theory,

$$(\mathrm{e}^{2\mathrm{i}\theta_i}) = (1,1,1,1,1,\mathrm{e}^{\mathrm{i}2\pi/3},\mathrm{e}^{\mathrm{i}2\pi/3},\mathrm{e}^{\mathrm{i}4\pi/3},\mathrm{e}^{\mathrm{i}4\pi/3})$$

which also the eigenvalues of T^2. However, the fermionic $\mathbb{Z}_3$ state cannot be the bosonic $\mathbb{Z}_3$ state described by the $K = \begin{pmatrix} 0 & 3 \\ 3 & 0 \end{pmatrix}$ $U(1)\times U(1)$ CS theory. This is because A^i for the bosonic $\mathbb{Z}_3$ state are all positive while A^1 and A^2 for the fermionic $\mathbb{Z}_3$ state are negative.

For the $K = \begin{pmatrix} 0 & 3 \\ 3 & 1 \end{pmatrix}$ $U(1)\times U(1)$ CS theory,

$$(\mathrm{e}^{2\mathrm{i}\theta_i}) = (1,1,1,\mathrm{e}^{\mathrm{i}\frac{4\pi}{9}},\mathrm{e}^{\mathrm{i}\frac{4\pi}{9}},\mathrm{e}^{\mathrm{i}\frac{10\pi}{9}},\mathrm{e}^{\mathrm{i}\frac{10\pi}{9}},\mathrm{e}^{\mathrm{i}\frac{16\pi}{9}},\mathrm{e}^{\mathrm{i}\frac{16\pi}{9}}).$$

For the $K = \begin{pmatrix} 0 & 3 \\ 3 & 2 \end{pmatrix}$ $U(1)\times U(1)$ CS theory,

$$(\mathrm{e}^{2\mathrm{i}\theta_i}) = (1,1,1,\mathrm{e}^{\mathrm{i}\frac{2\pi}{9}},\mathrm{e}^{\mathrm{i}\frac{2\pi}{9}},\mathrm{e}^{\mathrm{i}\frac{8\pi}{9}},\mathrm{e}^{\mathrm{i}\frac{8\pi}{9}},\mathrm{e}^{\mathrm{i}\frac{14\pi}{9}},\mathrm{e}^{\mathrm{i}\frac{14\pi}{9}}).$$

For the $K = \begin{pmatrix} 0 & 3 \\ 3 & 4 \end{pmatrix}$ $U(1)\times U(1)$ CS theory,

$$(\mathrm{e}^{2\mathrm{i}\theta_i}) = (1,1,1,\mathrm{e}^{\mathrm{i}\frac{4\pi}{9}},\mathrm{e}^{\mathrm{i}\frac{4\pi}{9}},\mathrm{e}^{\mathrm{i}\frac{10\pi}{9}},\mathrm{e}^{\mathrm{i}\frac{10\pi}{9}},\mathrm{e}^{\mathrm{i}\frac{16\pi}{9}},\mathrm{e}^{\mathrm{i}\frac{16\pi}{9}}).$$

For the $K = \begin{pmatrix} 0 & 3 \\ 3 & 5 \end{pmatrix}$ $U(1)\times U(1)$ CS theory,

$$(\mathrm{e}^{2\mathrm{i}\theta_i}) = (1,1,1,\mathrm{e}^{\mathrm{i}\frac{2\pi}{9}},\mathrm{e}^{\mathrm{i}\frac{2\pi}{9}},\mathrm{e}^{\mathrm{i}\frac{8\pi}{9}},\mathrm{e}^{\mathrm{i}\frac{8\pi}{9}},\mathrm{e}^{\mathrm{i}\frac{14\pi}{9}},\mathrm{e}^{\mathrm{i}\frac{14\pi}{9}}).$$

They are all different from the eigenspectrum of T^2. Therefore, our fermionic $\mathbb{Z}_3$ state is described by the $K = \begin{pmatrix} 0 & 3 \\ 3 & 3 \end{pmatrix} \sim \begin{pmatrix} 3 & 0 \\ 0 & -3 \end{pmatrix}$ $U(1)\times U(1)$ CS theory.

D. A class of "trivial" fermionic fixed points

There is a class of fermionic solutions that can be constructed from the bosonic solutions. For every bosonic solution $(N_{ijk}, N^f_{ijk} = 0, F^{ijm,\alpha\beta,a}_{kln,\gamma\lambda,b}, O^{jk,\alpha\beta}_{i,a}, A^i)$ that satisfy the conditions (91, 92, 93, 94), we can obtain a fermionic solution $(N_{ijk}, N^f_{ijk} = N_{ijk}, \tilde F^{ijm,\alpha\beta,a}_{kln,\gamma\lambda,b}, \tilde O^{jk,\alpha\beta}_{i,a}, \tilde A^i)$ that satisfy the same set of conditions. Here $(\tilde F^{ijm,\alpha\beta,a}_{kln,\gamma\lambda,b}, \tilde O^{jk,\alpha\beta}_{i,a}, \tilde A^i)$ are given by

$$\begin{aligned}\tilde F^{ijm,\alpha\beta,a}_{kln,\gamma\lambda,b} &= ab F^{ijm,\alpha\beta,a}_{kln,\gamma\lambda,b}\\ \tilde O^{jk,\alpha\beta}_{i,a} &= a O^{jk,\alpha\beta}_{i,a}, \quad \tilde A^i = A^i.\end{aligned} \tag{150}$$

For example, when $N = 0$ and $N_{000} = N^f_{000} = 1$, we obtain the following solution

$$\tilde F^{000,11,a}_{000,11,b} = ab, \quad \tilde O^{00,11}_{0,a} = a, \quad \tilde A^0 = 1. \tag{151}$$

Such a solution give rise to a fermionic fixed-point state which has one fermion on each vertex and no other dynamical degrees of freedom on the edges and vertices. This is a trivial fermionic state. We can view such a state as a fermionic "direct-product" state.

For every bosonic fixed-point state described by $(N_{ijk}, N^f_{ijk} = 0, F^{ijm,\alpha\beta,a}_{kln,\gamma\lambda,b}, O^{jk,\alpha\beta}_{i,a}, A^i)$, we can combined it with the above fermionic "direct-product" state. This leads to the corresponding fermionic fixed-point state described by $(N_{ijk}, N^f_{ijk} = N_{ijk}, \tilde F^{ijm,\alpha\beta,a}_{kln,\gamma\lambda,b}, \tilde O^{jk,\alpha\beta}_{i,a}, \tilde A^i)$. So, those fermionic fixed-point states are trivial in some sense.

IX. SUMMARY

Using string-net condensations and LU transformations, we have obtained a systematic understanding of a large class of topological orders in bosonic systems.[6,19] An interacting fermionic system is a non-local bosonic system. So classifying topological orders in fermion systems appears to be a very difficult problem.

In this paper, we introduced fLU and gfLU transformations, which allow us to develop a general theory for a large class of fermionic topological orders. We find that a large class of fermionic topological orders can be classified by the data $(N_{ijk}, N^f_{ijk}, F^{ijm,\alpha\beta,a}_{jkn,\chi\delta,b}, O^{jk,\alpha\beta}_{i,a})$ that satisfy a set of non-linear algebraic equations (91), (92), (93), and (94). Such a result generalizes the string-net result[6,19] to fermionic cases. If we impose more conditions such as the tetrahedral conditions (100), the orthogonality condition (102) as well as the associativity condition (119) into our theory, we can further generalize the famous Turaev-Viro invariance to its fermionic version. We believe such a generalization will enable us to describe all the quantum double of fermionic CS theory. We hope our approach is a starting point for a general theory of topological orders in interacting fermion systems.

We would like to thank M. Levin for some very helpful discussions. XGW is supported by NSF Grant No. DMR-1005541. ZCG is supported in part by the NSF Grant No. NSFPHY05-51164.

Appendix A: The fermionic structure of support space

To understand the fermionic structure of the support space $\tilde V_A$, let us first study the structure of ρ_A. Let $|\phi_i\rangle$ be a basis of the Hilbert space of the region A and $|\bar\phi_{\bar i}\rangle$ be a basis of the Hilbert space of the region outside of A. $|\psi\rangle$ can be expanded by $|\phi_i\rangle \otimes |\bar\phi_{\bar i}\rangle$:

$$|\psi\rangle = \sum_{i,\bar i} C_{i,\bar i} |\phi_i\rangle \otimes |\bar\phi_{\bar i}\rangle. \tag{A1}$$

Then the matrix elements of ρ_A is given by

$$(\rho_A)_{ij} = \sum_{\bar i} (C_{i\bar i})^* C_{j\bar i}. \tag{A2}$$

For a fermion system, the Hilbert space on a site, V_i has a structure: $V_i = V_i^0 \oplus V_i^1$, where states in V_i^0 have even numbers of fermions and states in V_i^1 have odd numbers of fermions. The Hilbert space on the region A, V_A, has a similar structure $V_A = V_A^0 \oplus V_A^1$, where states in V_A^0 have even number of fermions and states in V_A^1 have odd numbers of fermions. Let $|\phi_{i,\alpha}\rangle$ be a basis of V_A^α. Similarly, the Hilbert space on the region outside of A, $V_{\bar A}$, also has a structure $V_{\bar A} = V_{\bar A}^0 \oplus V_{\bar A}^1$. Let $|\bar\phi_{\bar i,\alpha}\rangle$ be a basis of $V_{\bar A}^\alpha$. In this case, $|\psi\rangle$ can be expanded as

$$|\psi\rangle = \sum_{i\alpha;\bar i\beta} C_{i,\alpha;\bar i,\beta} |\phi_{i,\alpha}\rangle \otimes |\bar\phi_{\bar i,\beta}\rangle. \tag{A3}$$

the matrix elements of ρ_A can now be expressed as

$$(\rho_A)_{i,\alpha;j,\beta} = \sum_{\bar i,\gamma} (C_{i,\alpha;\bar i,\gamma})^* C_{j,\beta;\bar i,\gamma}. \tag{A4}$$

Since the fermion number mod 2 is conserved, we may assume that $|\psi\rangle$ contains even numbers of fermions. This means $C_{i,\alpha;\bar i,\gamma} = 0$ when $\alpha + \gamma = 1 \bmod 2$. Hence, we find that

$$(\rho_A)_{i,\alpha;j,\beta} = 0, \text{ when } \alpha + \beta = 1 \bmod 2. \tag{A5}$$

Such a density matrix tells us that the support space $\tilde V_A$ has a structure $\tilde V_A = \tilde V_A^0 \oplus \tilde V_A^1$, where $\tilde V_A^0$ has even numbers of fermions and $\tilde V_A^1$ has odd numbers of fermions. This means that U_g contains only even numbers of fermionic operators (*ie* U_g is a pseudo-local bosonic operator).

Appendix B: Ideal Hamiltonian for the fixed-point states

In the section V, we have constructed the fixed-point wave functions from the solutions

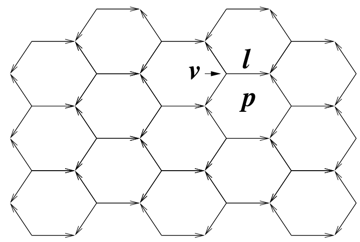


FIG. 13. A honeycomb lattice. The vertices are labeled by $\boldsymbol{v}$, hexagons by $\boldsymbol{p}$, and links by $\boldsymbol{l}$.

$(N_{ijk}, N^f_{ijk}, F^{ijm,\alpha\beta,a}_{kln,\gamma\lambda,b}, O^{jk,\alpha\beta}_{i,a}, A^i)$ of the self consistent conditions. In this section, we will show that those fixed-point wave functions on a honeycomb lattice (see Fig. 13) are exact gapped ground state of a local Hamiltonian

$$\hat{H} = \sum_{\boldsymbol{v}} (1 - \hat{Q}_{\boldsymbol{v}}) + \sum_{\boldsymbol{p}} (1 - \hat{B}_{\boldsymbol{p}}) \tag{B1}$$

where $\sum_{\boldsymbol{v}}$ sums over all vertices and $\sum_{\boldsymbol{p}}$ sums over all hexagons. We will first discuss the unprojective case where

$$\mathrm{e}^{\mathrm{i}\,\theta^{ab}_F}|_{a=\pm,b=\pm} = \mathrm{e}^{\mathrm{i}\,\theta_{O1}} = \mathrm{e}^{\mathrm{i}\,\theta_{O2}} = 1. \tag{B2}$$

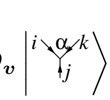
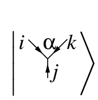
In this case, all the equal-up-to-total-phase relation $\simeq$ become the exactly-equal relation $=$.

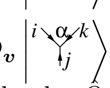
The Hamiltonian $\hat{H}$ should act on the Hilbert space V_G formed by all the graph states. It turns out that it is more convenient to write down the Hamiltonian if we expand the Hilbert space by adding an auxiliary qubit to each vertex:

$$V^{ex}_G = V_G \otimes (\otimes_{\boldsymbol{v}} V_{qubit}) \tag{B3}$$

where $\otimes_{\boldsymbol{v}}$ goes over all vertices and V_{qubit} is the two dimensional Hilbert space of an auxiliary qubit $|I\rangle$, $I = 0, 1$. So in the expanded Hilbert space V^{ex}_G, the states on a vertex $\boldsymbol{v}$ are labeled by $|\alpha\rangle \otimes |I\rangle$, $I = 0, 1$. V_G is embedded into V^{ex}_G in the following way: each vertex state $|\alpha\rangle$ in V_G correspond to the following vertex state $|\alpha\rangle \otimes |s_{ijk}(\alpha)\rangle$ in V^{ex}_G, where we have assume that the states on the three links connecting to the vertex are $|i\rangle$, $|j\rangle$, and $|k\rangle$. So the new auxiliary qubit $|I\rangle$ on a vertex is completely determined by (i, j, k, α) and does not represent an independent degree of freedom. It just tracks if the vertex state is bosonic or fermionic. The $|0\rangle$-state correspond to bosonic vertex states and the $|1\rangle$-state correspond to fermionic vertex states.

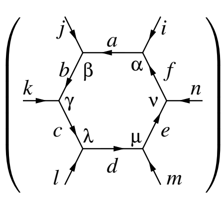
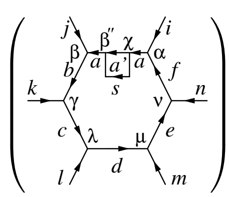
In the expanded Hilbert space, $\hat{Q}_{\boldsymbol{v}}$ in $\hat{H}$ acts on the states on the 3 links that connect to the vertex $\boldsymbol{v}$ and on the states $|\alpha\rangle \otimes |I\rangle$ on the vertex $\boldsymbol{v}$:

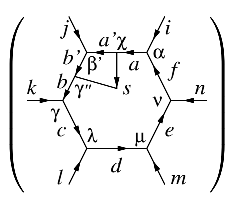
$\hat{Q}_{\boldsymbol{v}}$  $\otimes |I\rangle =$  $\otimes |I\rangle$ if $N_{ijk} > 0,\ I = s_{ijk}(\alpha)$,

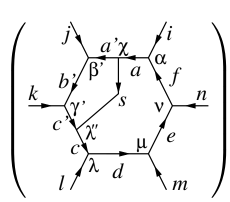
$\hat{Q}_{\boldsymbol{v}}$  $\otimes |I\rangle = 0$ otherwise. (B4)

Clearly, $\hat{Q}_{\boldsymbol{v}}$ is a projector $\hat{Q}^2_{\boldsymbol{v}} = \hat{Q}_{\boldsymbol{v}}$. The $\hat{B}_{\boldsymbol{p}}$ operator in $\hat{H}$ acts on the states on the 6 links and the 6 vertices of the hexagon $\boldsymbol{p}$ and on the 6 links that connect to the hexagon. However, $\hat{B}_{\boldsymbol{p}}$ operator will not alter the states on the 6 links that connect to the hexagon.

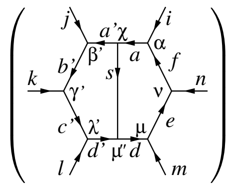
The $\hat{B}_{\boldsymbol{p}}$ operator is constructed from a combination of the F-moves and O-moves as shown below:

Ψ_{fix}  $= \sum_{sa'\chi\beta''} (\mathcal{O}^{sa',\chi\beta''}_{a,\underline{\chi\beta''}})^\dagger \Psi_{\text{fix}}$ 

$= \sum_{sa'\chi\beta''} \sum_{b'\gamma''\beta'} (\mathcal{O}^{sa',\chi\beta''}_{a,\underline{\chi\beta''}})^\dagger \mathcal{F}^{a'sa,\beta''\beta,\underline{\beta''\beta}}_{b^*j^*b'^*,\gamma''\beta',\underline{\gamma''\beta'}} \Psi_{\text{fix}}$ 

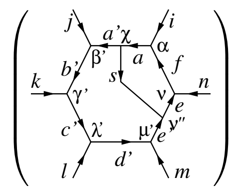
$= \sum_{sa'\chi\beta''} \sum_{b'\gamma''\beta'} \sum_{c'\lambda''\gamma'} (\mathcal{O}^{sa',\chi\beta''}_{a,\underline{\chi\beta''}})^\dagger \mathcal{F}^{a'sa,\beta''\beta,\underline{\beta''\beta}}_{b^*j^*b'^*,\gamma''\beta',\underline{\gamma''\beta'}} \mathcal{F}^{b'sb,\gamma''\gamma,\underline{\gamma''\gamma}}_{c^*k^*c'^*,\lambda''\gamma',\underline{\lambda''\gamma'}} \Psi_{\text{fix}}$ 

$$= \sum_{sa'\chi\beta''}\sum_{b'\gamma''\beta'}\sum_{c'\lambda''\gamma'}\sum_{d'\mu''\lambda'} (\mathcal{O}^{sa',\chi\beta''}_{a,\underline{\chi\beta''}})^\dagger \mathcal{F}^{a'sa,\beta''\beta,\underline{\beta''\beta}}_{b^*j^*b'^*,\gamma''\beta',\underline{\gamma''\beta'}} \mathcal{F}^{b'sb,\gamma''\gamma,\underline{\gamma''\gamma}}_{c^*k^*c'^*,\lambda''\gamma',\underline{\lambda''\gamma'}} \mathcal{F}^{c'sc,\lambda''\lambda,\underline{\lambda''\lambda}}_{d^*l^*d'^*,\mu''\lambda',\underline{\mu''\lambda'}} \Psi_{\rm fix}$$



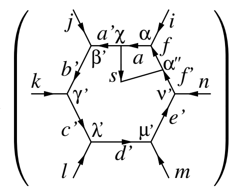
$$= \sum_{sa'\chi\beta''}\sum_{b'\gamma''\beta'}\sum_{c'\lambda''\gamma'}\sum_{d'\mu''\lambda'}\sum_{e'\nu''\mu'} (\mathcal{O}^{sa',\chi\beta''}_{a,\underline{\chi\beta''}})^\dagger \mathcal{F}^{a'sa,\beta''\beta,\underline{\beta''\beta}}_{b^*j^*b'^*,\gamma''\beta',\underline{\gamma''\beta'}} \mathcal{F}^{b'sb,\gamma''\gamma,\underline{\gamma''\gamma}}_{c^*k^*c'^*,\lambda''\gamma',\underline{\lambda''\gamma'}} \mathcal{F}^{c'sc,\lambda''\lambda,\underline{\lambda''\lambda}}_{d^*l^*d'^*,\mu''\lambda',\underline{\mu''\lambda'}} \times$$

$$\mathcal{F}^{d'sd,\mu''\mu,\underline{\mu''\mu}}_{e^*m^*e'^*,\nu''\mu',\underline{\nu''\mu'}} \Psi_{\rm fix}$$



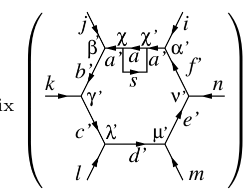
$$= \sum_{sa'\chi\beta''}\sum_{b'\gamma''\beta'}\sum_{c'\lambda''\gamma'}\sum_{d'\mu''\lambda'}\sum_{e'\nu''\mu'}\sum_{f'\alpha''\nu'} (\mathcal{O}^{sa',\chi\beta''}_{a,\underline{\chi\beta''}})^\dagger \mathcal{F}^{a'sa,\beta''\beta,\underline{\beta''\beta}}_{b^*j^*b'^*,\gamma''\beta',\underline{\gamma''\beta'}} \mathcal{F}^{b'sb,\gamma''\gamma,\underline{\gamma''\gamma}}_{c^*k^*c'^*,\lambda''\gamma',\underline{\lambda''\gamma'}} \mathcal{F}^{c'sc,\lambda''\lambda,\underline{\lambda''\lambda}}_{d^*l^*d'^*,\mu''\lambda',\underline{\mu''\lambda'}} \times$$

$$\mathcal{F}^{d'sd,\mu''\mu,\underline{\mu''\mu}}_{e^*m^*e'^*,\nu''\mu',\underline{\nu''\mu'}} \mathcal{F}^{e'se,\nu''\nu,\underline{\nu''\nu}}_{f^*n^*f'^*,\alpha''\nu',\underline{\alpha''\nu'}} \Psi_{\rm fix}$$



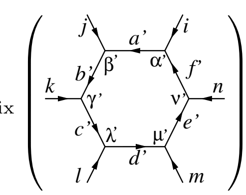
$$= \sum_{sa'\chi\beta''}\sum_{b'\gamma''\beta'}\sum_{c'\lambda''\gamma'}\sum_{d'\mu''\lambda'}\sum_{e'\nu''\mu'}\sum_{f'\alpha''\nu'}\sum_{\chi'\alpha'} (\mathcal{O}^{sa',\chi\beta''}_{a,\underline{\chi\beta''}})^\dagger \mathcal{F}^{a'sa,\beta''\beta,\underline{\beta''\beta}}_{b^*j^*b'^*,\gamma''\beta',\underline{\gamma''\beta'}} \mathcal{F}^{b'sb,\gamma''\gamma,\underline{\gamma''\gamma}}_{c^*k^*c'^*,\lambda''\gamma',\underline{\lambda''\gamma'}} \mathcal{F}^{c'sc,\lambda''\lambda,\underline{\lambda''\lambda}}_{d^*l^*d'^*,\mu''\lambda',\underline{\mu''\lambda'}} \times$$

$$\mathcal{F}^{d'sd,\mu''\mu,\underline{\mu''\mu}}_{e^*m^*e'^*,\nu''\mu',\underline{\nu''\mu'}} \mathcal{F}^{e'se,\nu''\nu,\underline{\nu''\nu}}_{f^*n^*f'^*,\alpha''\nu',\underline{\alpha''\nu'}} \mathcal{F}^{f'sf,\alpha''\alpha,\underline{\alpha''\alpha}}_{a^*i^*a'^*,\chi'\alpha',\underline{\chi'\alpha'}} \Psi_{\rm fix}$$

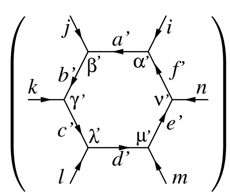


$$= \sum_{sa'\chi\beta''}\sum_{b'\gamma''\beta'}\sum_{c'\lambda''\gamma'}\sum_{d'\mu''\lambda'}\sum_{e'\nu''\mu'}\sum_{f'\alpha''\nu'}\sum_{\chi'\alpha'} (\mathcal{O}^{sa',\chi\beta''}_{a,\underline{\chi\beta''}})^\dagger \mathcal{F}^{a'sa,\beta''\beta,\underline{\beta''\beta}}_{b^*j^*b'^*,\gamma''\beta',\underline{\gamma''\beta'}} \mathcal{F}^{b'sb,\gamma''\gamma,\underline{\gamma''\gamma}}_{c^*k^*c'^*,\lambda''\gamma',\underline{\lambda''\gamma'}} \mathcal{F}^{c'sc,\lambda''\lambda,\underline{\lambda''\lambda}}_{d^*l^*d'^*,\mu''\lambda',\underline{\mu''\lambda'}} \times$$

$$\mathcal{F}^{d'sd,\mu''\mu,\underline{\mu''\mu}}_{e^*m^*e'^*,\nu''\mu',\underline{\nu''\mu'}} \mathcal{F}^{e'se,\nu''\nu,\underline{\nu''\nu}}_{f^*n^*f'^*,\alpha''\nu',\underline{\alpha''\nu'}} \mathcal{F}^{f'sf,\alpha''\alpha,\underline{\alpha''\alpha}}_{a^*i^*a'^*,\chi'\alpha',\underline{\chi'\alpha'}} \mathcal{O}^{s^*a,\chi'\chi}_{a',\underline{\chi'\chi}} \Psi_{\rm fix}$$



$$= \sum_{a'\alpha'}\sum_{b'\beta'}\sum_{c'\gamma'}\sum_{d'\lambda'}\sum_{e'\mu'}\sum_{f'\nu'} \mathcal{B}^{a\alpha,b\beta,c\gamma,d\lambda,e\mu,f\nu}_{a'\alpha',b'\beta',c'\gamma',d'\lambda',e'\mu',f'\nu'}(i,j,k,l,m,n)\Psi_{\rm fix} \tag{B5}$$



where the Majorana-number valued matrix $\mathcal{B}$ is given by

$$\mathcal{B}^{a\alpha,b\beta,c\gamma,d\lambda,e\mu,f\nu}_{a'\alpha',b'\beta',c'\gamma',d'\lambda',e'\mu',f'\nu'}(i,j,k,l,m,n) = \sum_{s\chi\beta''}\sum_{\gamma''}\sum_{\lambda''}\sum_{\mu''}\sum_{\nu''}\sum_{\alpha''}\sum_{\chi'} (\mathcal{O}^{sa',\chi\beta''}_{a,\underline{\chi\beta''}})^\dagger \mathcal{F}^{a'sa,\beta''\beta,\underline{\beta''\beta}}_{b^*j^*b'^*,\gamma''\beta',\underline{\gamma''\beta'}} \times$$

$$\mathcal{F}^{b'sb,\gamma''\gamma,\underline{\gamma''\gamma}}_{c^*k^*c'^*,\lambda''\gamma',\underline{\lambda''\gamma'}} \mathcal{F}^{c'sc,\lambda''\lambda,\underline{\lambda''\lambda}}_{d^*l^*d'^*,\mu''\lambda',\underline{\mu''\lambda'}} \mathcal{F}^{d'sd,\mu''\mu,\underline{\mu''\mu}}_{e^*m^*e'^*,\nu''\mu',\underline{\nu''\mu'}} \mathcal{F}^{e'se,\nu''\nu,\underline{\nu''\nu}}_{f^*n^*f'^*,\alpha''\nu',\underline{\alpha''\nu'}} \mathcal{F}^{f'sf,\alpha''\alpha,\underline{\alpha''\alpha}}_{a^*i^*a'^*,\chi'\alpha',\underline{\chi'\alpha'}} \mathcal{O}^{s^*a,\chi'\chi}_{a',\underline{\chi'\chi}} \tag{B6}$$

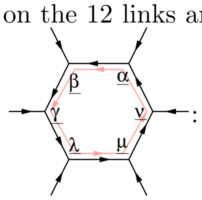
Now, let us choose a particular order of the vertices. We assume the two new created vertices $\underline{\beta''}$ and $\underline{\chi}$ satisfy $\underline{\chi} < \underline{\beta''} < \underline{\alpha}, \underline{\beta}, \underline{\gamma}, \underline{\lambda}, \underline{\mu}, \underline{\nu}$. When we do the six F-moves, we can choose $\underline{\beta'} = \underline{\beta}$, $\underline{\gamma'} = \underline{\gamma}$, $\underline{\lambda'} = \underline{\lambda}$, $\underline{\mu'} = \underline{\mu}$, $\underline{\nu'} = \underline{\nu}$, $\underline{\alpha'} = \underline{\alpha}$, and $\underline{\beta''} = \underline{\gamma''} = \underline{\lambda''} = \underline{\mu''} = \underline{\nu''} = \underline{\alpha''} = \underline{\chi'}$. For such a choice of ordering, the above becomes

$$
\begin{aligned}
&\mathcal{B}^{a\alpha,b\beta,c\gamma,d\lambda,e\mu,f\nu}_{a'\alpha',b'\beta',c'\gamma',d'\lambda',e'\mu',f'\nu'}(i,j,k,l,m,n) = \sum_{s\chi\beta"}\sum_{\gamma"}\sum_{\lambda"}\sum_{\mu"}\sum_{\nu"}\sum_{\alpha"}\sum_{\chi'} (\mathcal{O}^{sa',\chi\beta"}_{a,+})^\dagger \mathcal{F}^{a'sa,\beta"\beta,+}_{b^*j^*b'^*,\gamma"\beta',+} \times \\
&\quad \mathcal{F}^{b'sb,\gamma"\gamma,+}_{c^*k^*c'^*,\lambda"\gamma',+} \mathcal{F}^{c'sc,\lambda"\lambda,+}_{d^*l^*d'^*,\mu"\lambda',+} \mathcal{F}^{d'sd,\mu"\mu,+}_{e^*m^*e'^*,\nu"\mu',+} \mathcal{F}^{e'se,\nu"\nu,+}_{f^*n^*f'^*,\alpha"\nu',+} \mathcal{F}^{f'sf,\alpha"\alpha,+}_{a^*i^*a'^*,\chi'\alpha',+} \mathcal{O}^{s^*a,\chi'\chi}_{a',-} \\
&= \sum_{s\chi\beta"}\sum_{\gamma"}\sum_{\lambda"}\sum_{\mu"}\sum_{\nu"}\sum_{\alpha"}\sum_{\chi'} \\
&\quad (O^{a'^*s^*,\beta"\chi}_{a^*,-})^* \theta_{\underline{\beta}}^{s_{ajb^*}(\beta)} \theta_{\underline{\beta'}}^{s_{a'jb'^*}(\beta')} F^{a'sa,\beta"\beta,+}_{b^*j^*b'^*,\gamma"\beta',+} \theta_{\underline{\gamma}}^{s_{bkc^*}(\gamma)} \theta_{\underline{\gamma'}}^{s_{b'kc'^*}(\gamma')} F^{b'sb,\gamma"\gamma,+}_{c^*k^*c'^*,\lambda"\gamma',+} \theta_{\underline{\lambda}}^{s_{cld^*}(\lambda)} \theta_{\underline{\lambda'}}^{s_{c'ld'^*}(\lambda')} F^{c'sc,\lambda"\lambda,+}_{d^*l^*d'^*,\mu"\lambda',+} \times \\
&\quad \theta_{\underline{\mu}}^{s_{dme^*}(\mu)} \theta_{\underline{\mu'}}^{s_{d'me'^*}(\mu')} F^{d'sd,\mu"\mu,+}_{e^*m^*e'^*,\nu"\mu',+} \theta_{\underline{\nu}}^{s_{enf^*}(\nu)} \theta_{\underline{\nu'}}^{s_{e'nf'^*}(\nu')} F^{e'se,\nu"\nu,+}_{f^*n^*f'^*,\alpha"\nu',+} \theta_{\underline{\alpha}}^{s_{fia^*}(\alpha)} \theta_{\underline{\alpha'}}^{s_{f'ia'^*}(\alpha')} F^{f'sf,\alpha"\alpha,+}_{a^*i^*a'^*,\chi'\alpha',+} O^{s^*a,\chi'\chi}_{a',-}
\end{aligned}
\tag{B7}
$$

If we replace the two Majorana numbers $\theta_{\underline{\alpha}}$ and $\theta_{\underline{\alpha'}}$ by the same Majorana fermion operator $\hat{\eta}_{\underline{\alpha}}$ on vertex $\underline{\alpha}$, $\theta_{\underline{\beta}}$ and $\theta_{\underline{\beta'}}$ by $\hat{\eta}_{\underline{\beta}}$, *etc* , $\mathcal{B}$ will become the operator $\hat{B}_{\boldsymbol{p}}$ that we are looking for. Here, the Majorana fermion operator $\hat{\eta}_{\boldsymbol{v}}$ acts on the auxiliary qubit $|I\rangle$ on the vertex $\boldsymbol{v}$: $\hat{\eta}_{\boldsymbol{v}}|I\rangle = |1-I\rangle$. Or more precisely, after giving all the vertices an order, $\hat{\eta}_{\boldsymbol{v}}$ is given by

$$
\hat{\eta}_{\boldsymbol{v}} = \sigma^x_{\boldsymbol{v}} \prod_{\boldsymbol{v}'<\boldsymbol{v}} \sigma^z_{\boldsymbol{v}'}, \tag{B8}
$$

where $\sigma^l_{\boldsymbol{v}}$, $l = x, y, z$ are the Pauli matrices acting on the auxiliary qubit on the $\boldsymbol{v}$-vertex.

Now, we can express $\hat{B}_{\boldsymbol{p}}$ as a sum of products of operators which act on the 12 links and the 6 vertices of a (framed) hexagon :

$$
\hat{B}_{\boldsymbol{p}} = \sum_{s,\chi\beta"\gamma"\lambda"\mu"\nu"\alpha"\chi'} \hat{O}_1^{s,\beta"\chi} \hat{F}^{s,\beta"}_{\underline{\beta};\gamma"} \hat{F}^{s,\gamma"}_{\underline{\gamma};\lambda"} \hat{F}^{s,\lambda"}_{\underline{\lambda};\mu"} \hat{F}^{s,\mu"}_{\underline{\mu};\nu"} \hat{F}^{s,\nu"}_{\underline{\nu};\alpha"} \hat{F}^{s,\alpha"}_{\underline{\alpha};\chi'} \hat{O}_2^{s,\chi'\chi}, \tag{B9}
$$

where the $\hat{F}^{s,\beta"}_{\underline{\beta};\gamma"}$ operator, acting on the vertex $\underline{\beta}$ and its three legs, is given by

$$
\hat{F}^{s,\beta"}_{\underline{\beta};\gamma"} = \sum_{a'abb'j\beta\beta'} F^{a'sa,\beta"\beta,+}_{b^*j^*b'^*,\gamma"\beta',+} \times \left| \overset{\beta}{a\ {}_{j}\ b} \right\rangle \left\langle \overset{\beta'}{a'\ {}_{j}\ b'} \right| \hat{\eta}_{\underline{\beta}}^{s_{ajb^*}(\beta)+s_{a'jb'^*}(\beta')} \tag{B10}
$$

The operators $\hat{O}_1^{s,\beta"\chi}$ and $\hat{O}_2^{s,\chi'\chi}$ act on the $\underline{\alpha}$-$\underline{\beta}$ link. They are given by

$$
\hat{O}_1^{s,\beta"\chi} = \sum_{a'a} (O^{a'^*s^*,\beta"\chi}_{a^*,-})^* \left| \ a \ \right\rangle \left\langle \ a' \ \right| \tag{B11}
$$

and

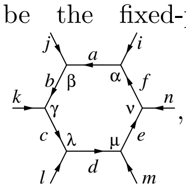
$$
\hat{O}_2^{s,\chi'\chi} = \sum_{a'a} O^{s^*a,\chi'\chi}_{a',-} \left| \ a \ \right\rangle \left\langle \ a' \ \right| \tag{B12}
$$

The operators $\hat{B}_{\boldsymbol{p}}$ and $\hat{F}^{s,\beta"}_{\underline{\beta};\gamma"}$ act on the expanded Hilbert space V_G^{sx} with auxiliary qubits. We can reduce them to act on the original Hilbert space V_G if we redefine the Majorana fermion operator $\hat{\eta}_{\boldsymbol{v}}$ to be

$$
\hat{\eta}_{\boldsymbol{v}} = \prod_{\boldsymbol{v}'<\boldsymbol{v}} \hat{\Sigma}_{\boldsymbol{v}'} \tag{B13}
$$

where

$$
\hat{\Sigma}_{\underline{\alpha}} = \sum_{ijk\alpha} (-)^{s_{ijk}(\alpha)} \left| {}^{i\ \alpha\ k}_{\quad j} \right\rangle \left\langle {}^{i\ \alpha\ k}_{\quad j} \right|. \tag{B14}
$$

In the following, we like to show that the matrix $\mathcal{B}$ is a projection matrix satisfying $\mathcal{B}^2 = \mathcal{B}$, which means that $\hat{B}_{\boldsymbol{p}}$ is a projector: $\hat{B}^2_{\boldsymbol{p}} = \hat{B}_{\boldsymbol{p}}$. First $\mathcal{B}^{a\alpha,b\beta,c\gamma,d\lambda,e\mu,f\nu}_{a'\alpha',b'\beta',c'\gamma',d'\lambda',e'\mu',f'\nu'}(i,j,k,l,m,n)$ can be viewed as an operator that act within the support space V_{ijklmn} of the fixed-point state Ψ_{fix} in the region surrounded by the fixed boundary states i, j, k, l, m, and n. Remember that the support space is defined s the following: Let $\Psi_{\text{fix}}(a,\alpha,b,\beta,c,\gamma,d,\lambda,e,\mu,f,\nu,i,j,k,l,m,n;\Gamma)$ be the fixed-point wave function on the graph , where Γ represent the indices on the other part of the graph which is not shown. If we fix i,j,k,l,m,n,Γ the wave function $\tilde{\Psi}_{i,j,k,l,m,n;\Gamma}(a,\alpha,b,\beta,c,\gamma,d,\lambda,e,\mu,f,\nu) = \Psi_{\text{fix}}(a,\alpha,b,\beta,c,\gamma,d,\lambda,e,\mu,f,\nu,i,j,k,l,m,n;\Gamma)$ can be viewed as a state in the support space V_{ijklmn}. As we vary Γ but keep i,j,k,l,m,n fixed, the wave functions $\tilde{\Psi}_{i,j,k,l,m,n;\Gamma}(a,\alpha,b,\beta,c,\gamma,d,\lambda,e,\mu,f,\nu)$ will span the full support space V_{ijklmn}. The relation (B5)

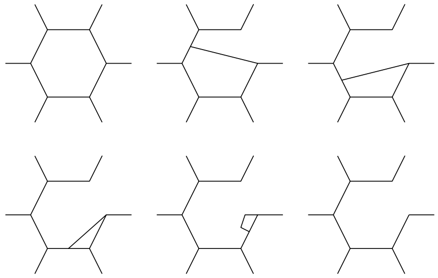
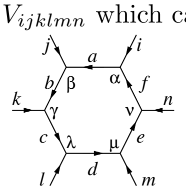
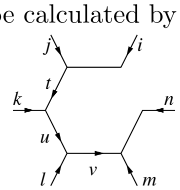
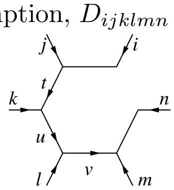


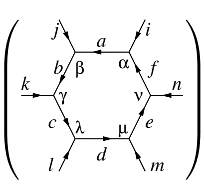
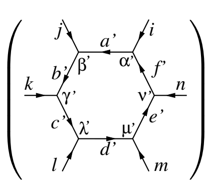
FIG. 14. $\mathcal{U}_P$ is generated by four F-moves and one O-move, which turn a hexagon in to a tree graph. $(\mathcal{U}_P)^\dagger$ is generated by one O-move and four F-moves, which run in the reversed direction.

implies that the operator represented by the matrix $\mathcal{B}^{a\alpha,b\beta,c\gamma,d\lambda,e\mu,f\nu}_{a'\alpha',b'\beta',c'\gamma',d'\lambda',e'\mu',f'\nu'}(i,j,k,l,m,n)$ is the identity operator in the support space V_{ijklmn}. However, the above discussion did not tell us how $\mathcal{B}$ acts on a state that is not in the support space.

To understand how $\mathcal{B}$ acts on a state that is not in the support space, let us consider the dimension D_{ijklmn} of the support space V_{ijklmn} which can be calculated by deforming the graph  into  through a gfLU transformation $\mathcal{U}$. Under the saturation assumption, D_{ijklmn} is equal to the distinct labels in the graph  (with i,j,k,l,m,n fixed):

$$D_{ijklmn} = \sum_{tuv} N_{ijt^*}N_{tku^*}N_{ulv^*}N_{vmn^*}. \qquad \text{(B15)}$$

We note that the operator $\mathcal{B}$ is realized by a combination of F-moves and O-moves (see eqn. (B5)) which realizes the following transformation

Ψ_{fix}  $\to \Psi_{\text{fix}}$ 

However, we can perform the same transformation via other combination of F-moves and O-moves. The self consistent conditions satisfied by the F-tensor and O-tensor ensure that all those different ways to transform between the two states lead to the same $\mathcal{B}$ matrix. In particular, we can have the following two paths that connect

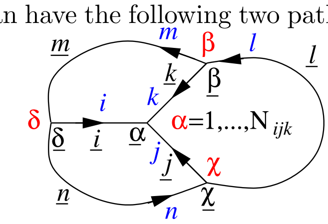
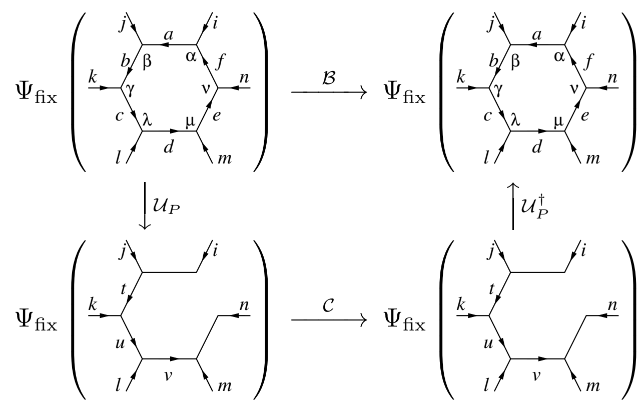


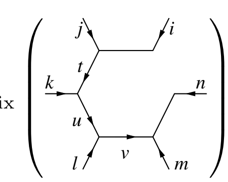
FIG. 15. A quantum state on a vertex-labeled graph G. The four vertices are labeled by $\underline{\alpha}$, $\underline{\beta}$, $\underline{\chi}$, and $\underline{\delta}$ and the six edges are labeled by $\underline{i}$, $\underline{j}$, $\underline{k}$, $\underline{l}$, $\underline{m}$, and $\underline{n}$. For example, $\underline{\alpha}=1$, $\underline{\beta}=2$, $\underline{\chi}=3$, $\underline{\delta}=4$, $\underline{i}=5$, $\underline{j}=6$, $\underline{k}=7$, $\underline{l}=8$, $\underline{m}=9$, $\underline{n}=10$ correspond to one vertex-edge labeling scheme. States on each edge are labeled by $l=0,...,N$. If the three edges of a vertex are in the states i, j, and k respectively, then the vertex has N_{ijk} states. The states on the $\underline{\alpha}$-vertex are labeled by $\alpha=1,...,N_{ijk}$. Note the orientation of the edges are point towards to vertex. Also note that $i\to j\to k$ runs anti-clockwise.

the two states (see Fig. 14):



where $\mathcal{U}$ is a gfLU transformation. We find that

$$\mathcal{B} = \mathcal{U}_P^\dagger \mathcal{C}\mathcal{U}_P \qquad \text{(B16)}$$

where $\mathcal{C}$, acting on Ψ_{fix} , is a dimension $D_{ijklmn}\times D_{ijklmn}$ identity matrix. Also $\mathcal{U}_P$, containing only one O-move (see Fig. 14), has a form $\mathcal{U}_P=\mathcal{U}_1\mathcal{P}\mathcal{U}_2$ where $\mathcal{U}_{1,2}$ are unitary matrices and $\mathcal{P}$ is a projection matrix. So the rank of $\mathcal{B}$ is equal or less than D_{ijklmn}. Since it is the identity in the D_{ijklmn}-dimensional space V_{ijklmn}, the matrix $\mathcal{B}$ is a hermitian projection matrix onto the space V_{ijklmn}:

$$\sum_{a''\alpha''}\sum_{b''\beta''}\sum_{c''\gamma''}\sum_{d''\lambda''}\sum_{e''\mu''}\sum_{f''\nu''}\mathcal{B}^{a\alpha,b\beta,c\gamma,d\lambda,e\mu,f\nu}_{a''\alpha'',b''\beta'',c''\gamma'',d''\lambda'',e''\mu'',f''\nu''}(i,j,k,l,m,n)\mathcal{B}^{a''\alpha'',b''\beta'',c''\gamma'',d''\lambda'',e''\mu'',f''\nu''}_{a'\alpha',b'\beta',c'\gamma',d'\lambda',e'\mu',f'\nu'}(i,j,k,l,m,n)$$
$$=\mathcal{B}^{a\alpha,b\beta,c\gamma,d\lambda,e\mu,f\nu}_{a'\alpha',b'\beta',c'\gamma',d'\lambda',e'\mu',f'\nu'}(i,j,k,l,m,n) \tag{B17}$$

In the above calculation of the $\mathcal{B}$, we first insert a bubble on the a-link. We may also calculate $\mathcal{B}$ by first inserting a bubble on other lines. All those different calculations will lead to the same $\mathcal{B}$ matrix, as discussed above.

We note that $\hat{B}_{\boldsymbol{p}_1}\hat{B}_{\boldsymbol{p}_2}$ and $\hat{B}_{\boldsymbol{p}_2}\hat{B}_{\boldsymbol{p}_1}$ are generated by different combinations of F-moves and O-moves. Since the two combinations transform between the same pair of states, they give rise to the same relation between the two states. Therefore $\hat{B}_{\boldsymbol{p}_1}$ and $\hat{B}_{\boldsymbol{p}_2}$ commute

$$\hat{B}_{\boldsymbol{p}_1}\hat{B}_{\boldsymbol{p}_2}=\hat{B}_{\boldsymbol{p}_2}\hat{B}_{\boldsymbol{p}_1}. \tag{B18}$$

We see that the corresponding Hamiltonian $\hat{H}$ is a sum of commuting projectors and is exactly soluble.

Now let us consider the projective cases where $(\mathrm{e}^{\mathrm{i}\theta_F^{ab}},\mathrm{e}^{\mathrm{i}\theta_{O1}},\mathrm{e}^{\mathrm{i}\theta_{O2}})$ are not all equal to one. Following the above discussion, we can show that $\hat{B}_{\boldsymbol{p}}\simeq(\mathcal{U}_P)^\dagger\mathcal{U}_P$. By adding a proper phase in $\hat{B}_{\boldsymbol{p}}$, we can make $\hat{B}_{\boldsymbol{p}}$ a hermitian projector $\hat{B}^2_{\boldsymbol{p}}=\hat{B}_{\boldsymbol{p}}=\hat{B}^\dagger_{\boldsymbol{p}}$. However, the $\hat{B}_{\boldsymbol{p}}$ operators on neighboring hexagons may not commute

$$\hat{B}_{\boldsymbol{p}_1}\hat{B}_{\boldsymbol{p}_2}=\mathrm{e}^{\mathrm{i}\theta_{\boldsymbol{p}_1\boldsymbol{p}_2}}\hat{B}_{\boldsymbol{p}_2}\hat{B}_{\boldsymbol{p}_1}. \tag{B19}$$

So if the phase factors $(\mathrm{e}^{\mathrm{i}\theta_F^{ab}},\mathrm{e}^{\mathrm{i}\theta_{O1}},\mathrm{e}^{\mathrm{i}\theta_{O2}})$ are not all equal to one (*ie* for the projective case), the corresponding fixed-point state is a ground state of a sum of projectors, which may not commute. Note that the projectors are unfrustrated and the ground state energy is zero.

Appendix C: Putting fermions on both vertices and edges

In the main body of the paper, we put fermions only on the vertices. Actually, the whole formulation for the fixed point wave functions can be generalized to systems with fermions on both vertices and edges. In this section, we will describe such an generalization.

1. Quantum state on a graph

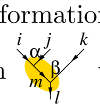
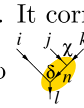
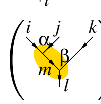
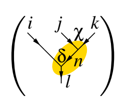
Here we also label the edges by $\underline{i}$, $\underline{j}$, *etc* (see Fig. 15). Now among the $N+1$ states on an edge, N^f of them are fermionic. We introduce s_i to indicate which state is fermionic:

$$\begin{aligned} &s_i=1\to|i\rangle \text{ is fermionic,}\\ &s_i=0\to|i\rangle \text{ is bosonic.}\end{aligned} \tag{C1}$$

Despite the similar look between i index and $\underline{i}$ index, and between α index and $\underline{\alpha}$ index, the two sets of indices are very different. For example, $\underline{\alpha}$ index labels the vertices while α index labels the states on a vertex. In this paper, we very often use α to label states on vertex $\underline{\alpha}$, and i to label states on edge $\underline{i}$.

In the string-net approach, we have assumed that the above graphic states on two graphs are the same if the two graphs have same topology. However, since different vertices and edges are really distinct, a generic graph state does not have such an topological invariance. Here we will consider vertex-edge-labeled graphs (ve-graphs) where each vertex and each edge are assigned an index $\underline{\alpha}$ and $\underline{i}$. Two ve-graphic is said to topologically the same if one graph can be continuously deformed into the other in such a way that vertex-edge labeling of the two graphs matches. We will consider the graph states that depend only on the topology of the ve-graphs.

2. The first type of wave function renormalization

The first type of renormalization does not change the degrees of freedom and corresponds to a local unitary transformation. It corresponds to locally deform the ve-graph  to . We note that the support space of ψ_{fix}  and ψ_{fix}  should have the same number of fermionic states. Thus eqn. (16) can be splitted as

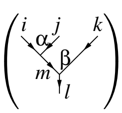
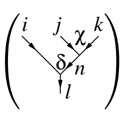
$$\begin{aligned}
&\sum_{m=0}^N(N^b_{jim^*}N^b_{kml^*}+N^f_{jim^*}N^f_{kml^*})(1-s_m)\\
&+\sum_{m=0}^N(N^b_{jim^*}N^f_{kml^*}+N^f_{jim^*}N^b_{kml^*})s_m\\
=&\sum_{n=0}^N(N^b_{kjn^*}N^b_{l^*ni}+N^f_{kjn^*}N^f_{l^*ni})(1-s_n),\\
&+\sum_{n=0}^N(N^b_{kjn^*}N^f_{l^*ni}+N^f_{kjn^*}N^b_{l^*ni})s_n.
\end{aligned} \tag{C2}$$

$$\sum_{m=0}^{N}(N^b_{jim^*}N^b_{kml^*}+N^f_{jim^*}N^f_{kml^*})s_m$$
$$+\sum_{m=0}^{N}(N^b_{jim^*}N^f_{kml^*}+N^f_{jim^*}N^b_{kml^*})(1-s_m)$$
$$=\sum_{n=0}^{N}(N^b_{kjn^*}N^b_{l^*ni}+N^f_{kjn^*}N^f_{l^*ni})s_n,$$
$$+\sum_{n=0}^{N}(N^b_{kjn^*}N^f_{l^*ni}+N^f_{kjn^*}N^b_{l^*ni})(1-s_n). \tag{C3}$$

We express the above unitary transformation in terms of the tensor $F^{ijm,\alpha\beta,a}_{kln,\chi\delta,b}$, where $a,b=\pm$, $i,j,k,...=0,...,N$, and $\alpha=1,...,N_{ijk}$, *etc* :

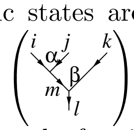
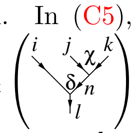
$$\phi_{ijkl,\Gamma}(\alpha,\beta,m)\simeq\sum_{n=0}^{N}\sum_{\chi=1}^{N_{kjn^*}}\sum_{\delta=1}^{N_{nil^*}}F^{ijm,\alpha\beta,\underline{\alpha\beta}}_{kln,\chi\delta,\underline{\chi\delta}}\tilde{\phi}_{ijkl,\Gamma}(\chi,\delta,n) \tag{C4}$$

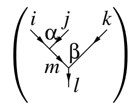
or graphically as

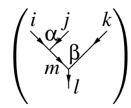
ψ_{fix}  $\simeq\sum_{n\chi\delta}F^{ijm,\alpha\beta,\underline{\alpha\beta}}_{kln,\chi\delta,\underline{\chi\delta}}\psi_{\text{fix}}$  . (C5)

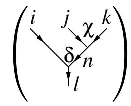
where the vertices carrying the states labeled by $(\alpha,\beta,\chi,\delta)$ are labeled by $(\underline{\alpha},\ \underline{\beta},\ \underline{\chi},\ \underline{\delta})$ and the edges carrying the states labeled by (i,j,k,l,m,n) are labeled by $(\underline{i},\ \underline{j},\ \underline{k},\ \underline{l},\ \underline{m},\ \underline{n})$ (see Fig. 15).

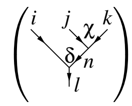
In general, we should write the F-tensor as $F^{ijm,\alpha\beta,\underline{\alpha},\underline{\beta},\underline{i},\underline{j},\underline{m}}_{kln,\chi\delta,\underline{\chi},\underline{\delta},\underline{k},\underline{l},\underline{n}}$ which depends the vertex and edge labels $\underline{\alpha}$, $\underline{\beta}$, $\underline{\chi}$, $\underline{\delta}$, $\underline{i}$, $\underline{j}$, $\underline{k}$, $\underline{l}$, $\underline{m}$, $\underline{n}$. Since the F-move only change the edge $\underline{m}$ to the edge $\underline{n}$. We can always set $\underline{m}=\underline{n}$. Since there is an unique canonical assignment of edge labels, we assume that the F-tensor does not depend on the edge labels $\underline{i}$, $\underline{j}$, $\underline{k}$, $\underline{l}$, $\underline{m}$, $\underline{n}$. As before, we also assume that the F-tensor only depends on the sign of $\underline{\beta}-\underline{\alpha}$. It does not depend on how big is the difference $|\underline{\beta}-\underline{\alpha}|$. So the F-tensor can be written as $F^{ijm,\alpha\beta,\underline{\alpha\beta}}_{kln,\chi\delta,\underline{\chi\delta}}$.

Since some values of $i,j,...,\alpha,\beta,...$ indices correspond to fermionic states, the sign of wave function depend on how those fermionic states are ordered. In (C5), the wave functions ψ_{fix}  and ψ_{fix}  are obtained by assuming the fermionic states are ordered in a particular way:

$\Big|\psi_{\text{fix}}$  $\Big\rangle$

$=\sum\psi^{\underline{\alpha}\,\underline{\beta}\,\underline{m}\,\underline{\eta}_1\,\underline{\eta}_2\cdots}_{\text{fix}}$  $|\alpha\beta m\eta_1\eta_2...\rangle_{\,\underline{\alpha}\,\underline{\beta}\,\underline{m}\,\underline{\eta}_1\,\underline{\eta}_2\cdots}$

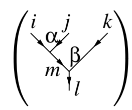
$\Big|\psi_{\text{fix}}$  $\Big\rangle$

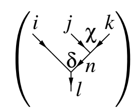
$=\sum\psi^{\underline{\chi}\,\underline{\delta}\,\underline{n}\,\underline{\eta}_1\,\underline{\eta}_2\cdots}_{\text{fix}}$  $|\chi\delta n\eta_1\eta_2...\rangle_{\,\underline{\chi}\,\underline{\delta}\,\underline{n}\,\underline{\eta}_1\,\underline{\eta}_2\cdots}$

(C6)

where $\sum$ sum over the indices on the vertices and on the edges and η_i are indices on other vertices.

Thus eqn. (C5) should be more properly written as

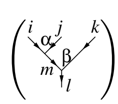
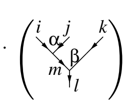
$\psi^{\underline{\alpha}\,\underline{\beta}\,\underline{m}\cdots}_{\text{fix}}$ 

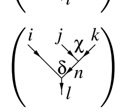
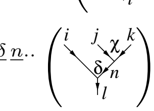
$\simeq\sum_{n=0}^{N}\sum_{\chi=1}^{N_{kjn^*}}\sum_{\delta=1}^{N_{nil^*}}F^{ijm,\alpha\beta,\underline{\alpha\beta}}_{kln,\chi\delta,\underline{\chi\delta}}\psi^{\underline{\chi}\,\underline{\delta}\,\underline{n}\cdots}_{\text{fix}}$  , (C7)

where the superscripts $\underline{\alpha}\,\underline{\beta}\,\underline{m}...$ and $\underline{\chi}\,\underline{\delta}\,\underline{n}...$ describing the order of fermionic states are added in the wave function.

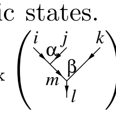
Since the sign of the wave function depend on the ordering of fermionic states, the F-tensor may also depend on the ordering. In this paper, we choose a particular ordering of fermionic states to define the F-tensor as described by $\underline{\alpha}\,\underline{\beta}...$ and $\underline{\chi}\,\underline{\delta}...$ in eqn. (C7). In such a canonical ordering, we create a fermion on the $\underline{\beta}$-vertex before we create a fermion on the $\underline{\alpha}$-vertex. Similarly, we create a fermion on the $\underline{\delta}$-vertex before we create a fermion on the $\underline{\chi}$-vertex.

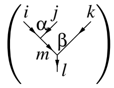
After introducing one Majorana numbers $\theta_{\underline{\alpha}}$ on each vertex $\underline{\alpha}$ and $\theta_{\underline{i}}$ on each edge $\underline{i}$, we can rewrite (C7) in a form that will be valid for any ordering of fermionic states on vertices and edges. We introduce the following wave function with Majorana numbers:

Ψ_{fix}  $=[\theta^{s_{jim^*}(\alpha)}_{\underline{\alpha}}\theta^{s_{kml^*}(\beta)}_{\underline{\beta}}\theta^{s_m}_{\underline{m}}..]\psi^{\underline{\alpha}\,\underline{\beta}\,\underline{m}..}_{\text{fix}}$ 

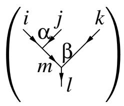
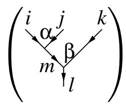
Ψ_{fix}  $=[\theta^{s_{kjn^*}(\chi)}_{\underline{\chi}}\theta^{s_{nil^*}(\delta)}_{\underline{\delta}}\theta^{s_n}_{\underline{n}}..]\psi^{\underline{\chi}\,\underline{\delta}\,\underline{n}..}_{\text{fix}}$ 

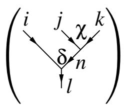
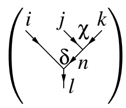
(C8)

where the order of the Majorana numbers $\theta_{\underline{\alpha}}\theta_{\underline{\beta}}\theta_{\underline{m}}...$ is tied to the order $\underline{\alpha}\,\underline{\beta}\,\underline{m}..$ in the superscript that describes the order of the fermionic states. We see that, by construction, the sign of Ψ_{fix}  does not depend on

the order of the fermionic states, and this is why the Majorana wave function Ψ_{fix}  does not carry the superscript $\underline{\alpha}\,\underline{\beta}\,\underline{m}\ldots$

We like to mention that $(\theta_{\underline{\alpha}}, \theta_{\underline{\beta}})$ and $(\theta_{\underline{\chi}}, \theta_{\underline{\delta}})$ are treated as different Majorana number even when, for example, $\underline{\alpha}$ and $\underline{\chi}$ take the same value. This is because $\underline{\alpha}$ and $\underline{\chi}$ label different vertices regardless if $\underline{\alpha}$ and $\underline{\chi}$ have the same value or not. So a more accurate notation should be

Ψ_{fix}  $= [\theta_{\underline{\alpha}}^{s_{jim^*}(\alpha)}\theta_{\underline{\beta}}^{s_{kml^*}(\beta)}\ldots]\psi_{\text{fix}}^{\underline{\alpha}\,\underline{\beta}\ldots}$ 

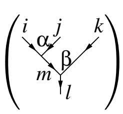
Ψ_{fix}  $= [\tilde{\theta}_{\underline{\chi}}^{s_{kjn^*}(\chi)}\tilde{\theta}_{\underline{\delta}}^{s_{nil^*}(\delta)}\ldots]\psi_{\text{fix}}^{\underline{\chi}\,\underline{\delta}\ldots}$  , (C9)

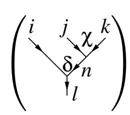
where $\theta_{\underline{\alpha}}$ and $\tilde{\theta}_{\underline{\chi}}$ are different even when $\underline{\alpha} = \underline{\chi}$. But in this paper, we will drop the ˜ and hope that it will not cause any confusions.

Let us introduce the F-tensor with Majorana numbers:

$$\begin{aligned}&\mathcal{F}^{ijm,\alpha\beta,\underline{\alpha\beta}}_{kln,\chi\delta,\underline{\chi\delta}} \qquad\qquad (C10)\\ &= \theta_{\underline{\alpha}}^{s_{jim^*}(\alpha)}\theta_{\underline{\beta}}^{s_{kml^*}(\beta)}\theta_{\underline{m}}^{s_m}\theta_{\underline{n}}^{s_n}\theta_{\delta}^{s_{nil^*}(\delta)}\theta_{\chi}^{s_{kjn^*}(\chi)}F^{ijm,\alpha\beta,\underline{\alpha\beta}}_{kln,\chi\delta,\underline{\chi\delta}}\end{aligned}$$

We can rewrite (C7) as

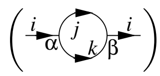
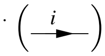
Ψ_{fix} 

$\simeq \sum_{n=0}^{N}\sum_{\chi=1}^{N_{kjn^*}}\sum_{\delta=1}^{N_{nil^*}}\mathcal{F}^{ijm,\alpha\beta,\underline{\alpha\beta}}_{kln,\chi\delta,\underline{\chi\delta}}\Psi_{\text{fix}}$  . (C11)

Such an expression is valid for any ordering of the fermion states.

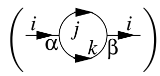
3. The second type of wave function renormalization

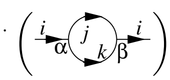
The second type renormalization can now be written as

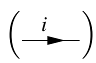
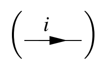
$\psi_{\text{fix}}^{\underline{\alpha}\,\underline{\beta}\,\underline{i_\alpha}\,\underline{i_\beta}\,\underline{j}\,\underline{k}\ldots}$  $\simeq O^{jk,\alpha\beta}_{i,\underline{\alpha\beta}}\psi_{\text{fix}}^{\underline{i}\ldots}$  . (C12)

where $\underline{i_\alpha}$ and $\underline{i_\beta}$ label the edges that connect to $\underline{\alpha}$ and $\underline{\beta}$ vertices respectively, and the ordering of the fermionic states on the vertices and the edges is described by $\underline{\alpha}\,\underline{\beta}\,\underline{i_\alpha}\,\underline{i_\beta}\,\underline{j}\,\underline{k}\ldots$. Note that the ordering of $\underline{j}$ and $\underline{k}$ is fixed by making $\underline{i_\beta} \to \underline{j} \to \underline{k}$ to run anti-clockwise.

The wave functions in eqn. (C12) is defined with respect to the ordering of the fermionic states described by $\underline{\alpha}\,\underline{\beta}\,\underline{i_\alpha}\,\underline{i_\beta}\,\underline{j}\,\underline{k}\ldots$. Let us introduce

Ψ_{fix}  $= [\theta_{\underline{\alpha}}^{s_{ik^*j^*}(\alpha)}\theta_{\underline{\beta}}^{s_{i^*jk}(\beta)}\theta_{\underline{i_\alpha}}^{s_i}\theta_{\underline{i_\beta}}^{s_i}\times$

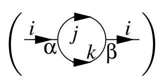
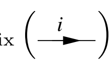
$\theta_{\underline{j}}^{s_j}\theta_{\underline{k}}^{s_k}\ldots]\psi_{\text{fix}}^{\underline{\alpha}\,\underline{\beta}\,\underline{i_\alpha}\,\underline{i_\beta}\,\underline{j}\,\underline{k}\ldots}$  ,

Ψ_{fix}  $= [\theta_{\underline{i}}^{s_i}\ldots]\psi_{\text{fix}}^{\underline{i}\ldots}$  (C13)

and

$$\mathcal{O}^{jk,\alpha\beta}_{i,\underline{\alpha\beta}} = \theta_{\underline{\alpha}}^{s_{ik^*j^*}(\alpha)}\theta_{\underline{\beta}}^{s_{i^*jk}(\beta)}\theta_{\underline{i_\alpha}}^{s_i}\theta_{\underline{i_\beta}}^{s_i}\theta_{\underline{j}}^{s_j}\theta_{\underline{k}}^{s_k}\theta_{\underline{i}}^{s_i}O^{jk,\alpha\beta}_{i,\underline{\alpha\beta}}. \qquad (C14)$$

We can rewrite (C12) as

Ψ_{fix}  $\simeq \mathcal{O}^{jk,\alpha\beta}_{i,\underline{\alpha\beta}}\Psi_{\text{fix}}$  (C15)

which is valid for any ordering of the fermionic states.

4. Fixed-point conditions on the F-tensor and the O-tesnor

We can derive the fixed-point conditions on the F-tensor and the O-tesnor in the similar as described before. The fixed-point conditions are a set of non-linear equations whose variables are N_{ijk}, N^f_{ijk}, s_i, $F^{ijm,\alpha\beta,a}_{kln,\gamma\lambda,b}$, $O^{jk,\alpha\beta}_{i,a}$, and the seven phases $(\mathrm{e}^{\mathrm{i}\theta_F^{ab}}|_{a=\pm,b=\pm}, \mathrm{e}^{\mathrm{i}\theta_{O1}}, \mathrm{e}^{\mathrm{i}\theta_{O2}}$. Here we just list the fixed-point conditions below

$$
\bullet\ s_i = s_{i^*}, \qquad \bullet\ N_{kji} = N_{i^*j^*k^*}, \qquad \bullet\ N^f_{kji} = N^f_{i^*j^*k^*}, \qquad \bullet\ \sum_{m=0}^{N} N_{jim^*} N_{kml^*} = \sum_{n=0}^{N} N_{kjn^*} N_{l^*ni},
$$

$$
\begin{aligned}
\bullet\ &\sum_{m=0}^{N} (N^b_{jim^*} N^b_{kml^*} + N^f_{jim^*} N^f_{kml^*}) s_m + \sum_{m=0}^{N} (N^b_{jim^*} N^f_{kml^*} + N^f_{jim^*} N^b_{kml^*})(1 - s_m) \\
&= \sum_{n=0}^{N} (N^b_{kjn^*} N^b_{l^*ni} + N^f_{kjn^*} N^f_{l^*ni}) s_n, + \sum_{n=0}^{N} (N^b_{kjn^*} N^f_{l^*ni} + N^f_{kjn^*} N^b_{l^*ni})(1 - s_n),
\end{aligned}
$$

$$
\bullet\ \sum_{jk} (N^b_{ik^*j^*} N^f_{i^*jk} + N^f_{ik^*j^*} N^b_{i^*jk}) \frac{1 - (-)^{s_i+s_j+s_k}}{2} + \sum_{jk} (N^b_{ik^*j^*} N^b_{i^*jk} + N^f_{ik^*j^*} N^f_{i^*jk}) \frac{1 + (-)^{s_i+s_j+s_k}}{2} \geq 1. \tag{C16}
$$

$$
\bullet\ \sum_{n\chi\delta} F^{ijm',\alpha'\beta',\,\underline{\alpha\beta}}_{kln,\chi\delta,\,\underline{\chi\delta}} (F^{ijm,\alpha\beta,\,\underline{\alpha\beta}}_{kln,\chi\delta,\,\underline{\chi\delta}})^* = \delta_{m\alpha\beta,m'\alpha'\beta'}, \qquad \bullet\ (F^{ijm,\alpha\beta,a}_{kln,\chi\delta,b})^* = (-)^{s_{jim^*}(\alpha)s_{kml^*}(\beta)} F^{jkn,\chi\delta,b}_{l^*i^*m^*,\beta\alpha,-a},
$$

$$
\bullet\ \sum_{n\chi\delta} F^{ijm,\alpha\beta,a}_{kln,\chi\delta,+} (F^{ijm',\alpha'\beta',b}_{kln,\chi\delta,+})^* = \mathrm{e}^{\mathrm{i}\,\theta^{ab}_F} \sum_{n\chi\delta} F^{ijm,\alpha\beta,a}_{kln,\chi\delta,-} (F^{ijm',\alpha'\beta',b}_{kln,\chi\delta,-})^*,
$$

$$
\begin{aligned}
\bullet\ F^{ijm,\alpha\beta,a}_{kln,\chi\delta,b} = 0 \text{ when }\ & N_{jim^*} < 1 \text{ or } N_{kml^*} < 1 \text{ or } N_{kjn^*} < 1 \text{ or } N_{nil^*} < 1, \text{ or} \\
& s_{jim^*}(\alpha) + s_{kml^*}(\beta) + s_{kjn^*}(\chi) + s_{nil^*}(\delta) + s_m + s_n = \text{odd},
\end{aligned}
$$

$$
\begin{aligned}
\bullet\ &\sum_{t\eta\varphi\kappa} (-)^{s_t[s_{tin^*}(\varphi)+s_{lts^*}(\kappa)]} F^{ijm,\alpha\beta,+}_{knt,\eta\varphi,-} F^{itn,\varphi\chi,+}_{lps,\kappa\gamma,-} F^{jkt,\eta\kappa,+}_{lsq,\delta\phi,-} \\
&= (-)^{s_{jim^*}(\alpha)s_{lkq^*}(\delta)+s_m[s_{lnp^*}(\chi)+s_n]+s_q[s_{sip^*}(\gamma)+s_s+s_m]} \sum_{\epsilon} F^{mkn,\beta\chi,+}_{lpq,\delta\epsilon,-} F^{ijm,\alpha\epsilon,+}_{qps,\phi\gamma,-}.
\end{aligned} \tag{C17}
$$

$$
\bullet\ \sum_{k,j} \sum_{\alpha=1}^{N_{kii^*}} \sum_{\beta=1}^{N_{j^*jk^*}} O^{jk,\alpha\beta}_{i,a} (O^{jk,\alpha\beta}_{i,a})^* = 1, \qquad \bullet\ O^{jk,\alpha\beta}_{i,-} = (-)^{s_{ik^*j^*}(\alpha)s_{jki^*}(\beta)+s_i+s_js_k} O^{k^*j^*,\beta\alpha}_{i^*,+},
$$

$$
\bullet\ O^{jk,\alpha\beta}_{i,a} = 0, \text{ if } N_{ik^*j^*} < 1, \text{ or } N_{i^*jk} < 1, \text{ or } s_{ik^*j^*}(\alpha) + s_{i^*jk}(\beta) + s_i + s_j + s_k = \text{odd},
$$

$$
\bullet\ \mathrm{e}^{\mathrm{i}\,\theta_{O2}} O^{k^*j^*,\beta\alpha}_{i^*,+} = (-)^{s_{ik^*j^*}(\alpha)s_{jki^*}(\beta)+s_js_k} \sum_{m\mu\nu;n\gamma\lambda} (-)^{s_{in^*j^*}(\gamma)s_{jni^*}(\lambda)+s_js_n} F^{ij^*k,\alpha\beta,-}_{jim,\nu\mu,+} F^{j^*jm,\nu\mu,+}_{i^*i^*n^*,\lambda\gamma,-} O^{jn,\gamma\lambda}_{i,+},
$$

$$
\begin{aligned}
\bullet\ &\mathrm{e}^{\mathrm{i}\,\theta_{O1}} O^{jm,\alpha\eta}_{i,+} \delta_{ip}\delta_{\beta\delta} = \sum_{n\chi\lambda\gamma} (-)^{s_{kjn^*}(\chi)[s_{nj^*k^*}(\gamma)+s_{im^*j^*}(\alpha)]} (-)^{s_i[1+s_j+s_m]+s_n[s_k+s_m]+s_j[s_k+s_m+s_n]} \times \\
&(-)^{s_{kil^*}(\beta)[s_i+s_k+s_m+s_n]} F^{mjp,\eta\beta,+}_{kln,\chi\lambda,-} F^{ij^*m,\alpha\lambda,+}_{nlk,\gamma\delta,-} O^{jn^*,\gamma\chi}_{k^*,+}, \qquad \text{for all } k,i,l \text{ satisfying } N_{kil^*} > 0.
\end{aligned} \tag{C18}
$$

Finding N_{ijk}, N^f_{ijk}, s_i, $F^{ijm,\alpha\beta,a}_{kln,\gamma\lambda,b}$, and $O^{jk,\alpha\beta}_{i,a}$ that satisfy such a set of non-linear equations corresponds to finding a fixed-point gfLU transformation that has a non-trivial fixed-point wave function. So the solutions $(N_{ijk}, N^f_{ijk}, s_i, F^{ijm,\alpha\beta,a}_{kln,\gamma\lambda,b}, O^{jk,\alpha\beta}_{i,a})$ give us a characterization of fermionic topological orders (and bosonic topological orders as a special case with $N^f_{ijk} = 0$ and $s_i = 0$).

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